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This dissertation focuses on the study of positive steady states to classes of nonlinear reaction diffusion (elliptic) systems on bounded domains as well as on exterior domains with Dirichlet boundary conditions. In particular, we study such systems in the challenging case when the reaction terms are negative at the origin, referred in the literature as semipositone problems.

For the last 30 years, study of elliptic partial differential equations with semipositone structure has flourished not only for the semilinear case but also for quasilinear case. Here we establish several results that directly contribute to and enhance the literature of semipositone problems. In particular, we discuss existence, non-existence and multiplicity results for classes of superlinear as well as sublinear systems. We establish our results via the method of sub-super solutions, degree theory arguments, a priori bounds and energy analysis.

POSITIVE SOLUTIONS OF NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS

by

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Approved by

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To my father

Negussie Abebe

APPROVAL PAGE

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CHAPTER I

INTRODUCTION

We consider nonlinear eigenvalue problems of the form

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in} \quad \Omega; \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{P_λ}$$

where $\lambda > 0$ is a parameter, $\Omega \subset \mathbb{R}^N$ $N \geq 1$ is a bounded domain with smooth boundary $\partial\Omega$ and $\Delta u := \text{div}(\nabla u)$ (The Laplacian operator). These problems arise in the study of steady state reactions diffusion processes. Studies of such models are of great importance in various applications in Physics, Biology, Chemistry, Engineering and in many other disciplines. For example in population dynamics see [BS79b], [BS79a], [BS82], [DT98], [MGHN92], [OSS02], [Sel98], [Sat72], in nonlinear heat generation and combustion theory see [Tam79], [Kel69], [Par74], [Ari69], [CD74] and [CD80], and in wave equations see [Str77]. The nonlinearity $f : [0, \infty) \to \mathbb{R}$ is the reaction term associated with the application in question. In the case when the nonlinearity satisfies $f(0) > 0$ and f monotonically increasing, (P_λ) is referred in the literature as *a positone* problem.

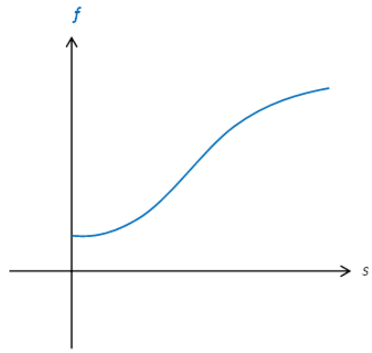

Figure 1. Positone f

Such type of problems were first initiated by Cohen and Keller [KC67]. Their study was motivated by the problem of temperature distribution in a body heated by a uniform electric current. The study of positive solutions to positone type problems has a rich history (spanning over 50 years). For review of literature and results of positone problems we refer to [Lio82], [Ama76], [BIS81] , [CR73], [dFLN82], [GNN79], [Lae71], [KW75], [KC67], [LSY09b], [CL70], [Sat73] and [Rab86] and the references in these papers.

When $f(0) < 0$ and eventually positive, (P_λ) is referred in the literature as *semi-positone* problem.

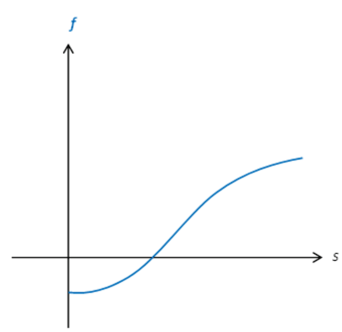

Figure 2. Semipositone f

Semipositone problems were first encountered by Brown and Shivaji [BS83] in the study of perturbed bifurcation theory, namely, in the study of positive solutions of the equation:

$$\begin{cases} -\Delta u = & \lambda(u - u^3) - \epsilon \qquad \text{in} \quad \Omega; \\ u = & 0 \qquad\qquad\qquad\quad \text{on} \quad \partial\Omega, \end{cases}$$

where $\epsilon > 0$ and $\lambda > 0$. In 1988, Castro and Shivaji [CS88] "formally" introduced semipositone problems as *nonpositone* problems. The mathematical analysis of positive solutions to semipositone problems is more challenging compared to the case of positone problems. This was confirmed in the celebrated SIAM Review paper by Lions [Lio82] and also by Berestycki-Caffarelli-Nirenberg in [BCN96]. In the last three decades, many researchers have made significant contributions to the study of positive solutions of (P_λ) with semipositone structure. Positive solutions of semipositone problems are delicate to study since the solutions have to live in regions where f is negative as well as in regions where f is positive (since we are seeking positive solutions). Semipositone problems arise naturally in ecological models when constant yield harvesting of species is involved. Recently there has been considerable efforts to extend the study of positive solutions of (P_λ) with semipositone structure in the following directions:

(a) Study coupled systems of quasilinear equations of the form:

$$\begin{cases} -\Delta_p u = \lambda_1 f_1(u) + \mu_1 g_1(v) =: f(u,v) & \text{in} \quad \Omega; \\ -\Delta_q v = \lambda_2 f_2(u) + \mu_2 g_2(v) =: g(u,v) & \text{in} \quad \Omega; \\ u = v = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $\lambda_i \geq 0$, $\mu_i \geq 0$ for $i = 1, 2$ are parameters. Here $\Delta_m Z := \text{div}(|\nabla Z|^{m-2}\nabla Z)$; $m > 1$ is the m-Laplacian operator, and when $m \neq 2$ we need to deal with this nonlinear diffusion operator in addition to the issues related to the semipositone structure of the reaction terms.

(b) Study coupled systems of semilinear equations of the form:

$$\begin{cases} -\Delta u = \lambda K_1(|x|) f(v) & \text{in} \quad \Omega_e; \\ -\Delta v = \lambda K_2(|x|) g(u) & \text{in} \quad \Omega_e; \\ u = v = 0 & \text{on} \quad \partial\Omega; \\ u(x) \to 0 \text{ and } v(x) \to 0 & \text{as} \quad |x| \to \infty, \end{cases}$$

where $\Omega_e := \mathbb{R}^N \backslash \overline{\Omega}$ is an exterior domain in $\mathbb{R}^N$ and $K_i(|x|) \to 0$ as $|x| \to \infty$ and $\lambda > 0$ is a parameter.

To build a mathematical framework to analyze such models one usually extends the reactions terms for negative arguments in a convenient way. However, since f and g are negative at the origin no extensions are possible to easily conclude that solutions to the extended problem are nonnegative componentwise and hence are nonnegative solutions of above coupled systems.

Now we discuss main results of this thesis.

1.1 Superlinear Elliptic Quasilinear Systems in a Ball

Results of this section are joint work with M. Chhetri (see [AC13]).

Consider a quasilinear systems of the form:

$$\begin{cases} -\Delta_p u = \lambda f(v) \quad \text{in} \quad \Omega; \\ -\Delta_p v = \lambda g(u) \quad \text{in} \quad \Omega; \\ \quad u = v = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a parameter, Ω is a unit ball in $\mathbb{R}^N$, $N \geq 1$, centered at the origin and $f, g : [0, \infty) \to \mathbb{R}$ are C^1 strictly increasing functions satisfying:

(H1) $f(0) < 0$, $g(0) < 0$ (semipositone), and

(H2) there exist α, $\beta \in (p-1, p*)$ such that

$$\lim_{s\to\infty} \frac{f(s)}{s^\alpha} > 0; \quad \lim_{s\to\infty} \frac{g(s)}{s^\beta} > 0, \tag{1.2}$$

where $p^* = \begin{cases} \frac{Np}{N-p}; & p < N \\ +\infty; & p \geq N \end{cases}$ is the critical Sobolev exponent.

Our first main result in this thesis is:

Theorem 1.1. *Suppose that (H1) and H(2) hold. Then there exists $\overline{\lambda} > 0$ such that* (1.1) *has no positive radially symmetric and radially decreasing solution for $\lambda > \overline{\lambda}$.*

Hypotheses of Theorem 1.1 is easily satisfied by $f(s) = s^a - \epsilon_1$ and $g(s) = s^b - \epsilon_2$ where $a, b > \max\{p-1, 1\}$, $\epsilon_1 > 0$ and $\epsilon_2 > 0$.

For the semilinear case, $p = 2$, such a nonexistence result for (1.1) was established in [DOS06]. For $p = 2$, it is also known that all nonnegative solutions are componentwise positive (see [CMS01]). Then it follows from [dF94] and [Tro81] that positive solutions are radially symmetric and radially decreasing. This enables the use of ordinary differential equation techniques. Note also that for $p = 2$, it is known that such systems have no positive solutions for λ large when Ω is any bounded domain in $\mathbb{R}^N$; $N > 1$ (see [CG09]).

Again when $p \neq 2$ but in the single equation case, using the result of [Bro98], it turns out that every nonnegative solution in a ball with nonlinearity f satisfying (H1) and (H2) is positive, radially symmetric and radially decreasing. Thus to study nonnegative solutions, it suffices to study positive solutions that are radially symmetric and radially decreasing. Using this result of [Bro98], nonexistence of nonnegative solution for large λ was established in a ball in [CG06]. For $p > 1, q = 2$ or $p, q \in (1, 2)$ an existence result for λ small has been recently established by Chhetri-Drábek -Shivaji in strictly convex bounded domains. (This manuscript is currently under review). When Ω is a ball, Theorem 1.1 complements this existence result with a non-existence result for λ large. However, we are restricted to proving the nonexistence of positive radially symmetric and radially decreasing solutions. When p or $q \neq 2$, to our knowledge it is an open question to establish that all nonnegative solutions are in fact positive, radially symmetric and radially decreasing.

1.2 Superlinear Elliptic Semilinear Problems on Exterior Domains

Results of this section are joint work with M. Chhetri, L. Sankar and R. Shivaji (see [ACSS14]).

We consider the following semilinear system:

$$\begin{cases} -\Delta u &= \lambda K_1(|x|)f(v) \quad \text{in } \Omega_e;\\ -\Delta v &= \lambda K_2(|x|)g(u) \quad \text{in } \Omega_e;\\ u(x) &= v(x) = 0 \quad \text{if } |x| = r_0(>0);\\ u(x) &\to 0, v(x) \to 0 \quad \text{as } |x| \to \infty, \end{cases} \tag{1.3}$$

where $\lambda > 0$ is a parameter and $\Omega_e = \{x \in \mathbb{R}^N \mid |x| > r_0, N > 2\}$ is an exterior domain. Here the nonlinearities $f, g : [0, \infty) \to \mathbb{R}$ are C^1 nondecreasing functions satisfying:

(A_1) $f(0) < 0$ and $g(0) < 0$ (semipositone), and

(A_2) for $i = 1, 2$ there exist $b_i > 0$ and $q_i > 1$ such that
$\lim\limits_{s\to\infty} \frac{f(s)}{s^{q_1}} = b_1$, and $\lim\limits_{s\to\infty} \frac{g(s)}{s^{q_2}} = b_2$.

Further, for $i = 1, 2$, the weight functions $K_i \in C^1([r_0, \infty), (0, \infty))$ are such that $K_i(|x|) \to 0$ as $|x| \to \infty$. In particular we are interested in the challenging case, where K_i do not decay too fast. Namely, we assume:

(A_3) There exist $\tilde{d}_1 > 0, \tilde{d}_2 > 0, \rho \in (0, N-2)$ such that for $i = 1, 2$

$$\frac{\tilde{d}_1}{|x|^{N+\rho}} \leq K_i(|x|) \leq \frac{\tilde{d}_2}{|x|^{N+\rho}} \quad \text{for} \quad |x| \gg 1\,.$$

We now state our second main result, namely:

Theorem 1.2. *Let $(A_1) - (A_3)$ hold. There exists $\overline{\lambda} > 0$ such that for $0 < \lambda < \overline{\lambda}$ system (1.3) has a positive radial solution. Moreover, $\|u\|_\infty \to \infty$ and $\|v\|_\infty \to \infty$ as $\lambda \to 0$.*

Nonlinearities $f(s) = b_1 s^a - \epsilon_1$ and $g(s) = b_2 s^b - \epsilon_2$, with $a, b > 1$, $\epsilon_1 > 0$ and $\epsilon_2 > 0$ satisfy the hypotheses of Theorem 1.2.

In the bounded domain case, such an existence result for single equations was established in [ANZ92], [AAB94], [CS88] and [Uns88]. This result was extended to systems case in [CG09] and [CG13]. Theorem 1.2 is the first result that establishes such an existence result for semipositone superlinear problems in an exterior domain. For recent results on semipositone problems with the reaction terms linear or sublinear at infinity, we refer to [HSS12], [LSY09a], [LSS11] and [SSS13] and references therein.

We also establish a nonexistence result for the single equation:

$$\begin{cases} -\Delta u = \lambda K_1(|x|)\tilde{f}(u) & \text{in } \Omega_e; \\ u(x) = 0 & \text{if } |x| = r_0 (> 0); \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{1.4}$$

for large values of λ, when $\tilde{f}, K_1$ satisfy the following hypotheses:

(A_4) $\tilde{f} : [0, \infty) \to \mathbb{R}$ is a C^1 function such that $\tilde{f}(0) < 0$, $\tilde{f}'(s) > 0$ for $s \geq 0$, and there exists $m_0 > 0$ such that $\lim\limits_{z \to \infty} \frac{f(z)}{z} \geq m_0$, and

(A_5) the weight function $K_1 \in C^1([r_0, \infty), (0, \infty))$ is such that $s^{\frac{-2(n-1)}{n-2}} K_1(r_0 s^{\frac{1}{2-n}})$ is decreasing for $s \in (0, 1]$.

Namely, our third result in this thesis is:

Theorem 1.3. *Let* (A_3)-(A_5) *hold. There exists* $\lambda^* > 0$ *such that* (1.4) *has no nonnegative radial solution for* $\lambda > \lambda^*$.

The function $\widetilde{f}(s) = s^a - \epsilon$ with $a > 1, \epsilon > 0$ and $K_1(r) = \frac{1}{r^{3+\rho}}$ for $\rho \in (0, 1)$ and $r \in [r_0, \infty)$ satisfy the hypothesis of Theorem 1.3.

As noted in Section 1.1, nonexistence results for such superlinear semipositone problems on bounded domain has a considerable history starting from the work in the eighties in [BCS89] to a recent work in [SY11]. However, Theorem 1.3 is the first such result for an exterior domain.

1.3 Sublinear Multiparameter Elliptic Quasilinear Systems

Theorem 1.4 and Theorem 1.5 of this section are joint work with M. Chhetri and R. Shivaji (see [ACS14]).

We consider quasilinear systems of the form:

$$\begin{cases} -\Delta_p u = \lambda_1 f_1(u) + \mu_1 \frac{g_1(v)}{v^{\alpha_1}} & \text{in} \quad \Omega; \\ -\Delta_q v = \lambda_2 \frac{f_2(u)}{u^{\alpha_2}} + \mu_2 g_2(v) & \text{in} \quad \Omega; \\ \quad u = v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^N$; $N \geq 1$,bounded domain with smooth boundary (a bounded interval if $N = 1$). For $i = 1, 2$, $0 \leq \alpha_i < 1$ are fixed constants and $\lambda_i, \mu_i \geq 0$ are parameters.

The nonlinearities f_i, $g_i : [0, \infty) \to \mathbb{R}$ are continuous functions such that $g_1(0) < 0$ and $f_2(0) < 0$. Let $\widetilde{g}_1(s) := \frac{g_1(s)}{s^{\alpha_1}}$ and $\widetilde{f}_2(s) := \frac{f_2(s)}{s^{\alpha_2}}$. We make the following assumptions:

(B1) $\widetilde{g}_1$ and $\widetilde{f}_2$ are nondecreasing,

(B2) $\lim\limits_{s\to\infty} f_1(s) = \lim\limits_{s\to\infty} g_2(s) = \lim\limits_{s\to\infty} \widetilde{g}_1(s) = \lim\limits_{s\to\infty} \widetilde{f}_2(s) = \infty$,

(B3) $\lim\limits_{s\to\infty} \frac{f_1(s)}{s^{p-1}} = \lim\limits_{s\to\infty} \frac{g_2(s)}{s^{q-1}} = 0$,

(B4) $\lim\limits_{s\to\infty} \frac{\widetilde{g}_1(M(\widetilde{f}_2(s))^{\frac{1}{q-1}})}{s^{p-1}} = 0$ for all $M > 0$.

Our fourth main result in this thesis is:

Theorem 1.4. *Let (B1)-(B4) hold. There exit $\nu_i > 0$ such that* (1.5) *has a positive solution when $\lambda_i + \mu_i > \nu_i$ for $i = 1, 2$.*

If $p = 2$, $q = 3$ and $\alpha_1 = \frac{2}{3}$ and $\alpha_2 = \frac{1}{3}$, it is easy to verify that $f_1(s) = s^{1/2} - \epsilon_1$; $g_1(s) = s - \epsilon_2$; $f_2(s) = s^{10/3} - \epsilon_3$ and $g_2(s) = s^{4/3} - \epsilon_4$ satisfy the hypotheses of Theorem 1.4, when ϵ_1, $\epsilon_4 \in \mathbb{R}$, $\epsilon_2 > 0$ and $\epsilon_3 > 0$.

The nonlinearities $\widetilde{g_1}$ and $\widetilde{f_2}$ are singular at 0 (since $\lim_{s\to 0^+} \tilde{g}_1(s) = \lim_{s\to 0^+} \tilde{f}_2(s) = -\infty$). The nonsingular case ($\alpha_i = 0$ for $i = 1, 2$) is well studied when the nonlinearities satisfy certain sublinear growth at infinity. For such results we refer to [AS07], [ANZ92], [AAB94], [CHS95], [HS03] and references therein. Recently, the singular case (α_1 or α_2 is non-zero) with sublinear behavior at infinity has also been studied in [LSY09a] and [LSY10]. However, in [LSY09a] the authors restrict their analysis to the case when $p = q$, $\alpha_1 = \alpha_2$ and $\lambda_1 = 0 = \mu_2$. In [LSY10] the authors study the system (1.5) again with $\lambda_1 = 0 = \mu_2$ but also restrict their analysis to the case when v^{α_1} is replaced by u^{α_1} and u^{α_2} is replaced by v^{α_2}. When the singularity occurs via the coupling components, as in (1.5), more delicate analysis is required to establish existence results, which we achieve in Theorem 1.4. Moreover, here we also allow λ_1 or μ_2 (or both) to be positive.

Next we consider the nonsingular case ($\alpha_1 = 0 = \alpha_2$), when f_i, $g_i : [0, \infty) \to \mathbb{R}$ are C^1 functions and satisfy:

(C1) $f_i(0) = 0 = g_i(0)$, for $i = 1, 2$, $f_1'(0) < 0$, $g_2'(0) < 0$ and $f_2'(0) = 0 = g_1'(0)$,

(C2) f_2 and g_1 are nondecreasing,

(C3) $\lim_{s\to\infty} f_i(s) = \lim_{s\to\infty} g_i(s) = \infty$ for $i = 1, 2$,

(C4) $\lim_{s\to\infty} \frac{f_1(s)}{s^{p-1}} = 0 = \lim_{s\to\infty} \frac{g_2(s)}{s^{q-1}}$, and

(C5) $\lim_{s\to\infty} \frac{g_1(M[f_2(s)]^{\frac{1}{q-1}})}{s^{p-1}} = 0$ for all $M > 0$.

We now state our fifth main result in this thesis, namely:

Theorem 1.5. *Suppose (C1)-(C5) hold. There exist* $\Theta_i > 0$ *such that* (1.5) *has two positive solutions when* $\lambda_i + \mu_i > \Theta_i$ *for* $i = 1, 2$.

For $p = 3$, $q = 4$, hypotheses of Theorem 1.5 are satisfied by
$f_1(s) = s^{3/2} - s$, $g_1(s) = s^{7/2}$, $f_2(s) = s^{3/2}$ and $g_2(s) = s^2 - 0.5s$ for $s \geq 0$.

The motivation for studying this multiplicity result of (1.5) comes from the work in [Rab74] for $p = 2$ and the single equation case, where the results were obtained using variational methods and degree-theoretic arguments. Also see [Per03] where such an approach was extended to the case $p > 1$ for the single equation case. Here we use the knowledge on semipositone problems combined with sub and supersolutions to establish our result, as in [AS07]. However, unlike in [AS07], here we allow some of the reaction terms to be negative near the origin. See also [MS99] where such a result was established for $p = 2$ in the single equation case via sub and supersolutions. Finally, we state the following nonexistence result when the parameters are small, which is our sixth main result in this thesis.

Theorem 1.6. *Let* $q = p$ *and let* $f_i, g_i : [0, \infty) \to \mathbb{R}$ *be continuous functions for* $i = 1, 2$. *Suppose there exist* $A_i, B_i > 0$ *such that* $f_1(s) \leq A_1 s^{p-1}$, $\widetilde{g_1}(s) \leq B_1 s^{p-1}$, $\widetilde{f_2}(s) \leq A_2 s^{p-1}$ *and* $g_2(s) \leq B_2 s^{p-1}$ *for* $s \geq 0$. *There exist* $\Lambda_i > 0$ *such that* (1.5) *has no nonnegative solution when* $\lambda_i + \mu_i < \Lambda_i$ *for* $i = 1, 2$.

The hypotheses of Theorem 1.6 are satisfied by
$f_1(s) = s^a - \epsilon_1$, $g_2(s) = s^b - \epsilon_2$ for $s \geq 0$ and $\widetilde{f_2}(s) = s^c - \frac{\epsilon_3}{s^\alpha}$, $\widetilde{g_1}(s) = s^d - \frac{\epsilon_4}{s^\beta}$ for $s > 0$, with $0 < a, b, c, d < p - 1$, $0 < \alpha, \beta < 1$ and $\epsilon_i > 0$, for $i = 1, 2, 3, 4$.

For the semilinear case, $p = q = 2$, such a nonexistence result was established in [DOS06]. Here we establish this result for the case $p = q > 1$. The case when $p \neq q$ remains open.

We conclude this dissertation with some computational results concerning positive solutions of the problems studied in this dissertation. We use shooting method to achieve these results. In particular, using Mathematica we generate bifurcation diagram for non-autonomous single equations and investigate positive solutions of coupled systems.

The outline of the rest of the dissertation is as follows. In Chapter II, we provide some preliminary results and discuss techniques (Leray-Schauder degree, radial forms, reduction via Kelvin transformations and the sub - super solutions method) that we will use to establish our results. Proof of Theorem 1.1 is presented in Chapter III, Proofs of Theorem 1.2 and Theorem 1.3 are presented in Chapter IV, Proofs of Theorem 1.4, Theorem 1.5, and Theorem 1.6 are presented in Chapter V, and our computational simulations in Chapter VI. In Chapter VII, we discuss some open problems we plan to focus in the near future. Lastly our Mathematica Codes are presented in the Appendix.

CHAPTER II

PRELIMINARIES

2.1 Degree Theory

Leray-Schauder degree is an important topological tool used in proving existence of solution to differential equations. In this section we first recall briefly the Brouwer degree in $\mathbb{R}^N$, its properties, and then discuss the Leray-Schauder degree in infinite dimensional spaces. In particular, we discuss the homotopy invariance property which we will use in our proof of Theorem 1.2. The discussion below is taken from [Llo78] and [FG95]. See [Maw99] for historical development of Leray-Schauder degree theory.

Brouwer degree in $\mathbb{R}^N$: Suppose $U \subset \mathbb{R}^N$ is an open and bounded set and let $\phi : \overline{U} \to \mathbb{R}^N$. If $\phi \in C(\bar{U}) \cap C^1(U)$ and $\underline{x} \in U$, we denote by $J_\phi(\underline{x})$, the determinant of the Jacobian matrix of ϕ at the point $\underline{x}$. Let $\underline{p} \in \mathbb{R}^N$ be a point such that $\phi \neq \underline{p}$ on ∂U and suppose $J_\phi(\underline{x}) \neq 0$ for all $\underline{x} \in \phi^{-1}(\underline{p}) \cap U$. Then it follows that $\phi^{-1}(\underline{p}) \cap U$ is finite (by virtue of Bolzano-Weierstrass theorem). The degree of ϕ at $\underline{p}$ relative to U, called Brouwer degree, is define by

$$d(\phi, U, \underline{p}) := \sum_{i=1}^{m} sgn(J_\phi(\underline{x}_i)),$$

where $\{\underline{x}_1, \underline{x}_2, ...\underline{x}_m\} = \phi^{-1}(\underline{p})$.

The degree, $d(\phi, U, \underline{p})$, of a continuous function $\phi \in C(\bar{U})$ is defined by approximating ϕ by C^1 functions.

Brouwer degree satisfies the following properties:

(1) **Additive property:** If $U = U_1 \cup U_2$, where $U_1, U_2 \subset \mathbb{R}^N$ are open bounded and disjoint, and if $\underline{p} \notin \phi(\partial U_1) \cup \phi(\partial U_2)$, then $d\left(\phi, U, \underline{p}\right) = d\left(\phi, U_1, \underline{p}\right) + d\left(\phi, U_2, \underline{p}\right)$.

(2) **Existence and normalization :** Let $U \subset \mathbb{R}^N$ be an open and bounded set. If $d\left(\phi, U, \underline{p}\right) \neq 0$, then $\underline{p} \in \phi(U)$ and $d(I, U, \underline{p}) = 1$, where I denotes the identity map of U.

(3) **Degree is continuous** in ϕ, U and $\underline{p}$.

(4) **Dependence on the boundary values:** If $\phi, \psi \in C(\bar{U}) \cap C^1(U)$ are such that $\phi|_{\partial U} = \psi|_{\partial U}$, then $d(\phi, U, \underline{p}) = d(\psi, U, \underline{p})$.

(5) **Excision:** Let $U \subset \mathbb{R}^N$ be an open and bounded, and $U_1 \subset U$ be such that $\underline{p} \notin \phi(\overline{U} \backslash U_1)$, then $d(\phi, U, \underline{p}) = d(\phi, U_1, \underline{p})$

(6) **Homotopy invariance:** Let $U \subset \mathbb{R}^N$ be an open and bounded set, and let $F : U \times [0,1] \to \mathbb{R}^N$ be continuous. If $F(x,t) \neq \underline{p}$ for each $(x,t) \in \partial U \times [0,1]$. Then $d\left(F(\cdot,t), U \times t, \underline{p}\right)$ is independent of t.

Leray-Schauder degree in infinite dimensional spaces: Let $(X, \|\cdot\|)$ be a normed linear space and $U \subset X$ be open and bounded set. Let $\phi := I - T$, where $T : \overline{U} \to X$ is compact. Let $\underline{p} \in X \backslash \phi(\partial U)$. It can be shown that there exists a continuous mapping $\widehat{T} : \overline{U} \to X$ with finite dimensional range such that $\|T(x) - \widehat{T}(x)\| < \text{dist}\left(\underline{p}, \phi(\partial\Omega)\right)$ for all $x \in \overline{U}$. Let $V := \text{span}\{T(\overline{U}), \underline{p}\}$, $U_V := \Omega \cap V$ and $\widehat{\phi} := I - \widehat{T}$. Then the Leray-Schauder degree, $d(\phi, U, \underline{p})$ is defined as, $d(\widehat{\phi}, U_V, \underline{p})$.

Leray-Schauder degree satisfies all the above properties (1)-(5), and the following homotopy invariance property.

Proposition 2.1. *[DMP03, 2.2.34-2.2.35] Let $(X, \|\cdot\|)$ be a normed linear space and let $U \subset X \times J$ be a bounded open set where J is a closed interval in $\mathbb{R}$. Let $U_t := \{\underline{x} \in X | (\underline{x}, t) \in U\}$, and let $(\partial U)_t := \{\underline{x} \in X | (\underline{x}, t) \in \partial U\}$. If $T : \overline{U} \to X$ is compact, and $\underline{p} \notin \phi((\partial U)_t, t)$ for all $t \in J$ where $\phi(\underline{x}, t) := x - T(\underline{x}, t)$, then $d(\phi_t, U_t, \underline{p}) =$ constant for $t \in J$ (here $\phi_t(\underline{x}) = \phi(\underline{x}, t)$).*

For a review of Leray-Schauder degree theory, see [FG95], [Llo78], and [Maw99].

2.2 Radial Form and Kelvin Transformation

Radial Forms

Let Ω be a unit ball in $\mathbb{R}^N$; $N \geq 1$ centered at the origin and (u, v) be radial solution of the system:

$$\begin{cases} -\Delta_p u = \lambda f(v) & \text{in } \Omega; \\ -\Delta_q v = \lambda g(u) & \text{in } \Omega; \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

We set $r = |x| = \sqrt{x_1^2 + x_2^2 + ... + x_N^2}$ and $w(r) = u(x)$. Then

$$\frac{\partial r}{\partial x_i} = \frac{2x_i}{2\sqrt{x_1^2 + x_2^2 + ... + x_N^2}} = \frac{x_i}{r} \quad \text{and} \quad \frac{\partial u}{\partial x_i} = \frac{dw(r)}{dr}\frac{\partial r}{\partial x_i} = w'(r)\frac{x_i}{r}.$$

Hence

$$\begin{aligned} |\nabla u| &= \sqrt{\sum_{i=1}^{N} \left(\tfrac{\partial u}{\partial x_i}\right)^2} = \sqrt{\sum_{i=1}^{N} \left(w'(r)\tfrac{x_i}{r}\right)^2} = |\tfrac{w'(r)}{r}|\sqrt{\sum_{i=1}^{N} (x_i)^2} \\ &= |w'(r)|. \end{aligned} \tag{2.2}$$

Then

$$
\begin{aligned}
\operatorname{div}(\nabla u) = \;& \sum_{i=1}^{N} \tfrac{\partial}{\partial x_i}\left(\tfrac{\partial u}{\partial x_i}\right) \\
=\;& \sum_{i=1}^{N} \tfrac{\partial}{\partial x_i}\left(w'(r)\tfrac{x_i}{r}\right) \\
=\;& \sum_{i=1}^{N} \left[w''(r)\tfrac{x_i}{r}\tfrac{x_i}{r} + \tfrac{w'(r)}{r} + w'(r)x_i(-\tfrac{1}{r^2})\tfrac{x_i}{r}\right] \\
=\;& w''(r) + \tfrac{N-1}{r}w'(r).
\end{aligned}
\tag{2.3}
$$

Also

$$
\begin{aligned}
\nabla(|\nabla u|^{p-2})\cdot\nabla u &= \sum_{i=1}^{N}\left(\tfrac{\partial}{\partial x_i}(|w'(r)|^{p-2})\cdot\tfrac{\partial u}{dx_i}\right) \\
&= \sum_{i=1}^{N}\left(\tfrac{d}{dr}(|w'(r)|^{p-2})\tfrac{\partial r}{\partial x_i}\cdot\tfrac{\partial u}{\partial x_i}\right) \\
&= \sum_{i=1}^{N}\left[\left((p-2)|w'(r)|^{p-3}\tfrac{w'(r)}{|w'(r)|}w''(r)\tfrac{x_i}{r}\right)\cdot\left(w'(r)\tfrac{x_i}{r}\right)\right] \\
&= \sum_{i=1}^{N}(p-2)|w'(r)|^{p-4}w''(r)\tfrac{1}{r^2}w'(r)x_i^2 \\
&= (p-2)|w'(r)|^{p-2}w''(r)\tfrac{1}{r^2}w'(r)\sum_{i=1}^{N}x_i^2 \\
&= (p-2)|w'(r)|^{p-2}w''(r).
\end{aligned}
\tag{2.4}
$$

Then using (2.2), (2.3) and (2.4) we compute

$$
\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \nabla(|\nabla u|^{\mathrm{p}-2})\cdot\nabla u + |\nabla u|^{\mathrm{p}-2}\operatorname{div}(\nabla u). \tag{2.5}
$$

Therefore

$$\begin{aligned}\Delta_p u &= (p-2)|w'(r)|^{p-2}w''(r) + |w'(r)|^{p-2}\left[w''(r) + \tfrac{N-1}{r}w'(r)\right] \\ &= (p-1)|w'(r)|^{p-2}w''(r) + \tfrac{N-1}{r}|w'(r)|^{p-2}w'(r)\,.\end{aligned} \tag{2.6}$$

Similarly, if we set $z(r) = v(x)$, we get

$$\Delta_q v = (q-1)|z'(r)|^{q-2}z''(r) + \tfrac{N-1}{r}|z'(r)|^{q-2}z'(r). \tag{2.7}$$

Hence studying radial solutions of (2.1) is equivalent to studying

$$\begin{cases} -(p-1)|w'(r)|^{p-2}w''(r) - \dfrac{N-1}{r}|w'(r)|^{p-2}w'(r) = \lambda f(z(r)), \quad 0 < r < 1; \\ -(q-1)|z'(r)|^{q-2}z''(r) - \dfrac{N-1}{r}|z'(r)|^{q-2}v'(r) = \lambda g(w(r)), \quad 0 < r < 1; \\ w(0) = w(1) = 0, \quad z(0) = z(1) = 0, \end{cases} \tag{2.8}$$

or, multiplying (2.8) by r^{N-1}, (2.1) is equivalent to the system:

$$\begin{cases} -(r^{N-1}\phi_p(w'(r)))' = \lambda r^{N-1} f(z(r)) \text{ for } \quad 0 < r < 1; \\ -(r^{N-1}\phi_q(z'(r)))' = \lambda r^{N-1} g(w(r)) \text{ for } \quad 0 < r < 1; \\ w'(0) = w(1) = 0; \\ z'(0) = z(1) = 0 \end{cases} \tag{2.9}$$

where $\phi_m(s) = |s|^{m-2}s$.

Kelvin Transformation

Consider

$$\begin{cases} -\Delta u &= \lambda K_1(|x|)f(v) \quad \text{in } \Omega_e; \\ -\Delta v &= \lambda K_2(|x|)g(u) \quad \text{in } \Omega_e; \\ u(x) &= v(x) = 0 \qquad \text{if } |x| = r_0(>0), \\ u(x) &\to 0, v(x) \to 0 \qquad \text{as } |x| \to \infty, \end{cases} \tag{2.10}$$

where $\lambda > 0$ is a parameter, and $\Omega_e = \{x \in \mathbb{R}^N \mid |x| > r_0, N > 2\}$ is an exterior domain. From our discussion earlier, studying radial solutions $(u(r), v(r))$ of (2.10) is equivalent to studying the system:

$$\begin{cases} -(r^{N-1}u'(r))' = \lambda r^{N-1}K_1(r)f(v(r)) \quad \text{for } r > r_0; \\ -(r^{N-1}v'(r))' = \lambda r^{N-1}K_2(r)g(u(r)) \quad \text{for } r > r_0; \\ u(r_0) = v(r_0) = 0, \\ u(r) \to 0, v(r) \to 0 \qquad \text{as } r \to \infty. \end{cases} \tag{2.11}$$

Now set $s = (\frac{r}{r_0})^{2-N}$, $w(s) = u(r)$ and $z(s) = v(r)$.

Then $r = r_0 s^{\frac{1}{2-N}}$ and $\frac{ds}{dr} = (2-N)\left(\frac{1}{r_0}\right)^{2-N} r^{1-N}$, and

$$
\begin{aligned}
|u'| = \;& |\tfrac{d}{dr} u(r)| \\
= \;& |\tfrac{d}{dr} w(s)| \\
= \;& |\tfrac{d}{ds} w(s) \tfrac{ds}{dr}| \\
= \;& |w'(s)||\tfrac{ds}{dr}| \\
= \;& |w'(s)|(N-2)(\tfrac{1}{r_0})^{N-2} r^{1-N}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&-\tfrac{d}{dr}\left[r^{N-1}u'(r)\right] \\
&\quad = -\tfrac{d}{dr}\left[r^{N-1}(2-N)w'(s)(\tfrac{1}{r_0})^{2-N} r^{1-N}\right] \\
&\quad = -(2-N)(\tfrac{1}{r_0})^{2-N} \tfrac{d}{ds}\left[w'(s)\right] \tfrac{ds}{dr} \\
&\quad = (2-N)(\tfrac{1}{r_0})^{2-N} w''(s)(2-N)(\tfrac{1}{r_0})^{2-N}(r_0 s^{\frac{1}{2-N}})^{1-N} \\
&\quad = -(2-N)^2(\tfrac{1}{r_0})^{2(2-N)-1+N} w''(s) s^{\frac{1-N}{2-N}} \\
&\quad = -(N-2)^2(\tfrac{1}{r_0})^{3-N} w''(s) s^{\frac{1-N}{2-N}}.
\end{aligned}
$$

Then the first equation in (2.11) becomes

$$
-(N-2)^2(\frac{1}{r_0})^{3-N} s^{\frac{1-N}{2-N}} w''(s) = \lambda(r_0 s^{\frac{1}{2-N}})^{N-1} K_1(r_0 s^{\frac{1}{2-N}}) f(z(s))
$$

which yields

$$\begin{aligned}
-w''(s) = {} & \lambda(N-2)^{-2} r_0^{3-N} s^{\frac{1-N}{N-2}} r_0^{N-1}\ s^{\frac{(1)(N-1)}{2-N}} K_1(r_0 s^{\frac{1}{2-N}}) f(z) \\
= {} & \lambda \tfrac{r_0^2}{(N-2)^2}\ s^{\frac{(2)(N-1)}{N-2}} K_1(r_0 s^{\frac{1}{2-N}}) f(z) \\
= {} & \lambda h_1(s) f(z),
\end{aligned}$$

where $h_1(s) := \frac{r_0^2}{(N-2)^2} s^{\frac{(2)(N-1)}{N-2}} K_1(r_0 s^{\frac{1}{2-N}})$. Similarly

$$-z''(s) = \lambda h_2(s) g(w(s))$$

where $h_2(s) := \frac{r_0^2}{(N-2)^2} s^{\frac{(2)(N-1)}{N-2}} K_2(r_0 s^{\frac{1}{2-N}})$.

Therefore studying radial solutions of (2.10) is equivalent to studying

$$\begin{cases}
-w''(s) = \lambda h_1(s) f(z(s)) & \text{for } 0 < s < 1; \\
-z''(s) = \lambda h_2(s) g(w(s)) & \text{for } 0 < s < 1; \\
w(0) = w(1) = 0, \quad z(0) = z(1) = 0. &
\end{cases}$$

2.3 Sub - Super Solutions Methods

One of the tools used in studying positive solutions to semipositone problems is a monotone iteration technique referred as the sub - super solutions method. See [Pao92] where this method is discussed for both parabolic and elliptic partial differential equations for single equation case as well as for system of equations. The difficulty of applying this method to semipositone problems is on finding a nonnegative subsolution. Consider an equation of the form:

$$\begin{cases} -\Delta u = & f(u) \quad \text{in} \quad \Omega; \\ u = & 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{2.12}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain and $f : [0, \infty) \to \mathbb{R}$ is a C^1 function.

A function $\underline{u} \in C^2(\Omega) \cap C(\overline{\Omega})$ is called a subsolution to (2.12) if

$$\begin{cases} -\Delta \underline{u} \leq & f(\underline{u}) \quad \text{in} \quad \Omega; \\ \underline{u} \leq & 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

A function $\overline{u} \in C^2(\Omega) \cap C(\overline{\Omega})$ is called a supersolution to (2.12) if the reversed inequalities are satisfied above. It is well known that if $\underline{u} \leq \overline{u}$, then (2.12) has a solution $u \in [\underline{u}, \overline{u}]$. In case of a positone problem (when $f(0) \geq 0$), clearly $\underline{u} = 0$ is a subsolution.

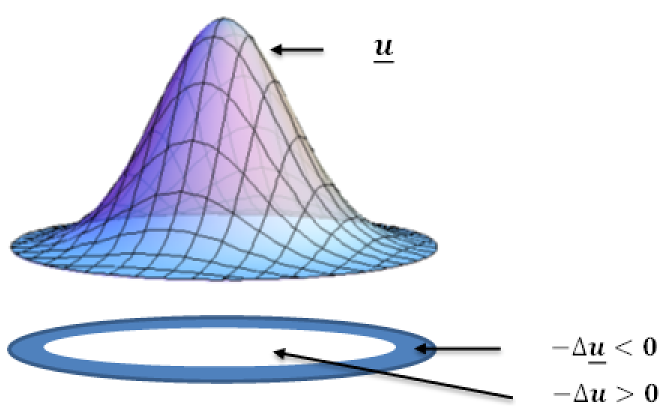

Figure 3. A subsolution

However to find a candidate for a nonnegative subsolution $\underline{u}$ for semipositone problems, we must consider functions $\underline{u}$ such that $-\Delta\underline{u} < 0$ near $\partial\Omega$ while $-\Delta\underline{u} > 0$ in a large part of the interior of Ω. Here we also consider the more even challenging case of infinite semipositone problems (when $\lim\limits_{s\to 0^+} f(s) = -\infty$) and in this case we need to consider $\underline{u}$ such that $-\Delta\underline{u} \to -\infty$ as $x \to \partial\Omega$.

Now we provide a formal definition of sub and supersolutions for the system

$$\begin{cases} -\Delta_p u = \lambda_1 f_1(u) + \mu_1 \frac{g_1(v)}{v^{\alpha_1}} =: H_1(u,v) & \text{in} \quad \Omega; \\ -\Delta_q v = \lambda_2 \frac{f_2(u)}{u^{\alpha_2}} + \mu_2 g_2(v) =: H_2(u,v) & \text{in} \quad \Omega; \\ u = v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.13}$$

where $p, q > 1$ and for $i = 1, 2;$ $\alpha_i \in [0, 1)$ are fixed constants and $\lambda_i, \mu_i > 0$ are parameters. For $i = 1, 2;$ $f_i,\, g_i \,:\, [0, \infty) \to \mathbb{R}$ are continuous functions such that $\widetilde{g}_1(s) := \frac{g_1(s)}{s^{\alpha_1}}$ and $\widetilde{f}_2(s) := \frac{f_2(s)}{s^{\alpha_2}}$ are nondecreasing for $s > 0$.

A subsolution of (2.13) is a pair of functions
$(\underline{u}, \underline{v}) \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega}) \times W_0^{1,q}(\Omega) \cap C(\overline{\Omega})$ satisfying $\underline{u} > 0$, $\underline{v} > 0$ in Ω and

$$\int_\Omega |\nabla\underline{u}|^{p-2}\nabla\underline{u} \cdot \nabla\xi dx \leq \int_\Omega H_1(\underline{u}, \underline{v})\xi dx$$

and

$$\int_\Omega |\nabla\underline{v}|^{q-2}\nabla\underline{v} \cdot \nabla\xi dx \leq \int_\Omega H_2(\underline{u}, \underline{v})\xi dx$$

for all $\xi \in W := \{\nu \in C_0^\infty(\Omega) \mid \nu \geq 0 \ \ in \ \ \Omega\}$.

A supersolution $(\overline{u}, \overline{v}) \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega}) \times W_0^{1,q}(\Omega) \cap C(\overline{\Omega})$ with $\overline{u} > 0$, $\overline{v} > 0$ in Ω is defined by reversing the above two inequalities.

It is known that (see [LSY09a]) if $(\underline{u}, \underline{v})$ is a subsolution and $(\overline{u}, \overline{v})$ is a supersolution with $(\underline{u}, \underline{v}) \leq (\overline{u}, \overline{v})$ componentwise, then (2.13) has a solution $(u, v) \in [(\underline{u}, \underline{v}), (\overline{u}, \overline{v})]$ with $(u, v) \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega}) \times W_0^{1,q}(\Omega) \cap C(\overline{\Omega})$. Here $(u_1, v_1) \leq (u_2, v_2)$ if $u_1 \leq u_2$ and $v_1 \leq v_2$ for all $x \in \Omega$.

In the nonsingular case ($\alpha_1 = 0 = \alpha_2$), there is no requirement that the sub and supersolutions have to be strictly positive in Ω. For this case the following three solution result also holds due to [Ama76] and [Shi87].

Proposition 2.2. *Suppose there exist a subsolution* $(\underline{u}, \underline{v})$*, a strict subsolution* $(\underline{u}_s, \underline{v}_s)$*, a strict supersolution* $(\overline{u}_s, \overline{v}_s)$ *and a supersolution* $(\overline{u}, \overline{v})$ *of* (1.5) *such that* $(\underline{u}, \underline{v}) \leq (\overline{u}_s, \overline{v}_s) \leq (\overline{u}, \overline{v})$, $(\underline{u}, \underline{v}) \leq (\overline{u}_s, \overline{v}_s) \leq (\overline{u}, \overline{v})$ *and* $(\underline{u}_s, \underline{v}_s) \nleq (\overline{u}_s, \overline{v}_s)$*. Then* (1.5) *has at least three distinct solutions* (u_1, v_1), (u_2, v_2), (u_3, v_3) *such that* $(u_1, v_1) \in [(\underline{u}, \underline{v}), (\overline{u}_s, \overline{v}_s)]$, $(u_2, v_2) \in [(\underline{u}_s, \underline{v}_s), (\overline{u}, \overline{v})]$
and $(u_3, v_3) \in [(\underline{u}, \underline{v}), (\overline{u}, \overline{v})] \setminus \{[(\underline{u}, \underline{v}), (\overline{u}_s, \overline{v}_s)] \cup [(\underline{u}_s, \underline{v}_s), (\overline{u}, \overline{v})]\}$*. By a strict sub (super) solution we mean a sub (super) solution that is not a solution.*

CHAPTER III

PROOF THEOREM 1.1

As described in Section 2.2., studying positive solutions of (1.1) is equivalent to studying the positive radial solutions of

$$\begin{cases} -\left(r^{N-1}\phi_p(u')\right)' = \lambda r^{N-1} f(v) \text{ for } \quad 0 < r < 1; \\ -\left(r^{N-1}\phi_p(v')\right)' = \lambda r^{N-1} g(u) \text{ for } \quad 0 < r < 1; \\ u'(0) = u(1) = 0; \\ v'(0) = v(1) = 0, \end{cases} \tag{3.1}$$

where $\phi_p(s) := |s|^{p-2}s$ for $s \neq 0$ and $\phi_p(0) = 0$. Clearly ϕ_p is an odd increasing homeomorphism of $\mathbb{R}$ onto itself. The inverse mapping of ϕ_p, denoted by $(\phi_p)_{-1}$, is given by $(\phi_p)_{-1} = \phi_{p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover ϕ_p is differentiable and its derivative, denoted by ϕ_p', is given by $\phi_p'(s) = (p-1)|s|^{p-2}$ for $s \neq 0$ and, $\phi_p'(0) = 0$ provided $p > 2$.

First we establish the following lemmas which will be crucial in proving our result.

3.1 Crucial Lemmas

Since we assume that positive solutions (u, v) of (3.1) are radially decreasing, we have $u'(r) < 0, \quad v'(r) < 0 \quad \text{on} \quad (0, 1]$. Then clearly any positive solution (u, v) of (3.1) satisfies $u(0) > 0, v(0) > 0$. It follows from (H1) and (H2) that f and g have unique positive zeros, say v_0 and u_0 respectively.

Define $F(t) := \int_0^t f(s)ds$ and $G(t) := \int_0^t g(s)ds$, and let V_0 and U_0 be the unique positive zeros of F and G respectively.

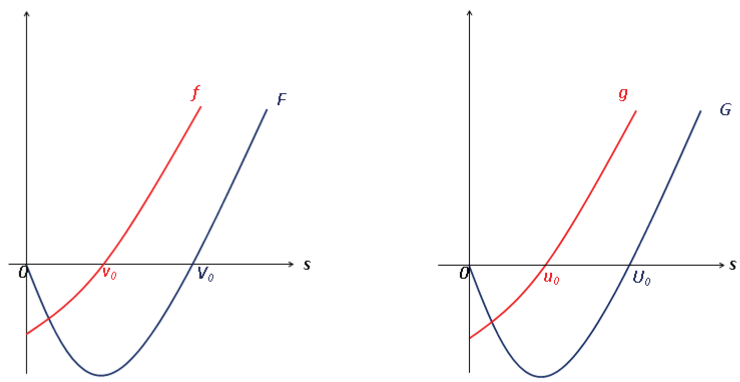

Figure 4. f, g and their primitives F and G

We observe that $0 < v_0 < V_0$ and $0 < u_0 < U_0$ (see Figure 4).

Lemma 3.1. *For any positive solution* (u, v) *of* (3.1), *we have* $u(0) > u_0$ *and* $v(0) > v_0$.

Proof: Assume to the contrary that $u(0) \leq u_0$ or $v(0) \leq v_0$. Without loss of generality, suppose that $v(0) \leq v_0$. Since $v(r)$ is radially decreasing, $v(r) < v(0) \leq v_0$ on $(0, 1)$ and thus $f(v(r)) < 0$. Then u satisfies

$$\begin{cases} -(r^{N-1}\phi_p(u')' = \lambda r^{N-1} f(v) < 0 \text{ for } \quad 0 < r < 1; \\ u'(0) = u(1) = 0\,. \end{cases} \tag{3.2}$$

Then the maximum principle for the scalar equation (see [CT00]), (3.2), yields that $u(r) \leq 0$, a contradiction since (u, v) is a positive solution. Therefore $u(0) > u_0$ and $v(0) > v_0$. ■

Now by setting

$$U_* := \min\{U_0, u(0)\} \quad \text{and} \quad V_* := \min\{V_0, v(0)\},$$

it follows from (H2) that there exists $K > 0$ such that

$$f(s) \geq K s^{\alpha} \quad \text{for all} \quad s > \frac{v_0 + V_*}{2}, \tag{3.3}$$

$$g(s) \geq K s^{\beta} \quad \text{for all} \quad s > \frac{u_0 + U_*}{2}. \tag{3.4}$$

Now we prove the following result in the interior of the ball.

Lemma 3.2. *Let (u, v) be a positive solution of (3.1). Then there exists $\lambda^* > 0$ such that for all $\lambda > \lambda^*$, there exist $r_1 = r_1(\lambda)$, $\widetilde{r_1} = \widetilde{r_1}(\lambda) \in \left[0, \frac{1}{2}\right]$ satisfying*

$$u(r_1) \;=\; \frac{u_0 + U_*}{2} \quad \textit{and} \quad v(\widetilde{r_1}) \;=\; \frac{v_0 + V_*}{2}.$$

Further, $|u'(r_1)|$, $|v'(\widetilde{r_1})| \to \infty$ as $\lambda \to \infty$.

Proof: First observe that since (u, v) is a positive solutions of (3.1), we have that $u(0) \geq U_* \geq \frac{u_0+U_*}{2}$ and $v(0) \geq V_* \geq \frac{v_0+V_*}{2}$. Now, without loss of generality, assume to the contrary that $v(r) > \frac{v_0+V_*}{2}$ for all $r \in [0, 1/2]$. Then u must satisfy:

(i) $u(r) > \frac{u_0+U_*}{2}$ for all $r \in [0, 1/2]$, or

(ii) $u(r_1) = \frac{u_0+U_*}{2}$ for some $r_1 \in [0, 1/2]$.

We will show that both of these cases lead to a contradiction.

Case 1: Suppose that $u(r) > \frac{u_0+U_*}{2}$ and $v(r) > \frac{v_0+V_*}{2}$ for all $r \in [0, 1/2]$. Integrating the first equation of (3.1) from 0 to $r \in (0, 1/2]$, using (H2) and the fact that $v' < 0$, we get

$$\begin{aligned} r^{N-1}\phi_p(u'(r)) &= -\lambda \int_0^r t^{N-1} f(v(t))dt \\ &\leq -\lambda K \int_0^r v^\alpha(t) t^{N-1} dt \\ &= -\frac{\lambda K}{N}\left[t^N v^\alpha(t)\right]_0^r + \frac{\lambda K \alpha}{N}\int_0^r v^{\alpha-1}(t)v'(t)t^N\,dt \\ &= -\frac{\lambda K}{N} r^N v^\alpha(r) + \frac{\lambda K \alpha}{N}\int_0^r v^{\alpha-1}(t)v'(t)t^N\,dt \\ &< -\frac{\lambda K}{N} r^N v^\alpha(r). \end{aligned} \tag{3.5}$$

Simplifying, applying the inverse of ϕ_p to the previous inequality and using the fact that ϕ_p is odd, we have

$$\begin{aligned} u'(r) &< \phi_{p'}\left(-\frac{\lambda r K v^\alpha(r)}{N}\right) \\ &= -\phi_{p'}\left(\frac{\lambda r K v^\alpha(r)}{N}\right) \\ &= -\left(\frac{\lambda r K}{N}\right)^{p'-1} v(r)^{\alpha(p'-1)} \\ &= -\left(\frac{\lambda r K}{N}\right)^{\frac{1}{p-1}} v(r)^{\frac{\alpha}{p-1}}. \end{aligned} \tag{3.6}$$

The last equality holds since $\frac{1}{p} + \frac{1}{p'} = 1$. Since $\frac{\alpha}{p-1} > 1$, we obtain

$$u'(r) < -\left(\frac{\lambda r K}{N}\right)^{\frac{1}{p-1}} v(r). \tag{3.7}$$

Similarly, using the second equation of (3.1) yields

$$v'(r) < -\left(\frac{\lambda r K}{N}\right)^{\frac{1}{p-1}} u(r). \tag{3.8}$$

Combining (3.7) and (3.8), we get

$$\frac{(u+v)'(r)}{(u+v)(r)} \quad < \quad -\left(\frac{\lambda r K}{N}\right)^{\frac{1}{p-1}} \qquad r \in [0, 1/2].$$

Integrating the above inequality from 0 to 1/4, we get

$$ln\left(\frac{u(\frac{1}{4})+v(\frac{1}{4})}{u(0)+v(0)}\right) = \int_0^{\frac{1}{4}} \frac{(u+v)'(r)}{(u+v)(r)} \; dr < -\left(\frac{\lambda K}{N}\right)^{\frac{1}{p-1}} \int_0^{\frac{1}{4}} r^{\frac{1}{p-1}} \,\mathrm{d}r = -\lambda^{\frac{1}{p-1}} C_0$$

where $C_0 = (\frac{K}{N})^{\frac{1}{p-1}} \int_0^{\frac{1}{4}} r^{\frac{1}{p-1}} \,\mathrm{d}r \quad > \quad 0$. Therefore

$$u(1/4) + v(1/4) \quad \leq \quad [u(0) + v(0)] e^{-\lambda^{\frac{1}{p-1}} C_0}.$$

This in turn implies that by choosing λ large, the expression $u(1/4) + v(1/4)$ can be made as small as desired. In particular, since (u, v) is a positive solution, there exists $\lambda^* > 0$ such that for $\lambda > \lambda^*$, $v(1/4) < \frac{v_0+V_*}{2}$, a contradiction.

Case 2: Suppose that $u(r_1) = \frac{u_0+U_*}{2}$ for some $r_1 \in [0, 1/2]$ and $v(r) > \frac{v_0+V_*}{2}$ for all $r \in [0, 1/2]$. Integrating the first equation of (3.1) from 0 to $r \in (0, 1/2]$, as in Case 1, we get

$$u'(r) < -\left(\frac{\lambda K}{N}\right)^{1/p-1} r^{1/(p-1)}.$$

Now integrating again from 0 to r_1, above inequality yields

$$u(r_1) - u(0) < -\left(\frac{\lambda K}{N}\right)^{1/(p-1)} \int_o^{r_1} r^{1/(p-1)}\, dr\,.$$

This in turn implies that there exists $\lambda^* > 0$ such that for $\lambda > \lambda^*$

$$u(r_1) < u(0) - \left(\frac{\lambda K}{N}\right)^{1/(p-1)} \int_o^{r_1} r^{1/(p-1)}\, dr < \frac{u_0 + U_*}{4},$$

a contradiction since $u(r_1) = \frac{u_0+U_*}{2}$.

Finally, it follows from (3.5) that

$$|u'(r_1)| \geq \left|\frac{\lambda K v^{\alpha}(r_1)}{N r_1}\right|^{\frac{1}{p-1}} \to \infty \quad \text{as} \quad \lambda \to \infty,$$

due to the fact that $0 < r_1 \leq \frac{1}{2}$ and hence $v(r_1) \not\to 0$. Similarly $|v'(\widetilde{r_1})| \to \infty$ as $\lambda \to \infty$. This proves the lemma. ∎

Lemma 3.3. *Let (u, v) be a positive solution of (3.1) and $c, \widetilde{c} > 2$ be any fixed constants. Then there exists $\lambda^{**} > 0$ such that for all $\lambda > \lambda^{**}$, there exist $r_2 = r_2(\lambda)$, $\widetilde{r_2} = \widetilde{r_2}(\lambda) \in \left[\frac{3}{4}, 1\right)$ satisfying*

$$u(r_2) \;=\; \frac{u_0}{c} \quad \textit{and} \quad v(\widetilde{r_2}) \;=\; \frac{v_0}{\widetilde{c}}.$$

Proof: Suppose that the lemma is false and let $c, \widetilde{c} > 2$. Then there exists a sequence $\{\lambda_n\}_n$ with $\lambda_n \to \infty$ as $n \to \infty$ and a corresponding sequence of positive solutions $\{(u_{\lambda_n}, v_{\lambda_n})\}_n$ of (3.1) such that either $u_{\lambda_n}(r) \neq \frac{u_0}{c}$ for all $r \in [3/4, 1)$ or $v_{\lambda_n}(r) \neq \frac{v_0}{\widetilde{c}}$ for all $r \in [3/4, 1)$ for all $n \in \mathbb{N}$. Assume, without loss of generality, that $u_{\lambda_n}(r) \neq \frac{u_0}{c}$ for all $r \in [3/4, 1)$ and for all n. Then we need to analyze the following two cases:

Case 1: $u_{\lambda_n}(r) \neq \frac{u_0}{c}$ and $v_{\lambda_n}(r) \neq \frac{v_0}{\widetilde{c}}$ for all $r \in [3/4, 1)$. Since u_{λ_n} is continuous, we observe that either $u_{\lambda_n}(r) > \frac{u_0}{c}$ for $r \in [3/4, 1)$ or $u_{\lambda_n}(r) < \frac{u_0}{c}$ for $r \in [3/4, 1)$. But the boundary condition $u_{\lambda_n}(1) = 0$ implies that we must have $u_{\lambda_n}(r) < \frac{u_0}{c}$ on $[3/4, 1)$. Similar argument yields $v_{\lambda_n}(r) < \frac{v_0}{\widetilde{c}}$ on $[3/4, 1)$.

Integrating the first equation of (3.1) from r to 1, for $r \in (3/4, 1)$, we obtain

$$r^{N-1}\phi_p(u'_{\lambda_n}(r)) = \phi_p(u'_{\lambda_n}(1)) + \lambda_n \int_r^1 s^{N-1} f(v_{\lambda_n}(s))ds.$$

Since $v_{\lambda_n} < v_o/\widetilde{c} < v_0/2$, f is nondecreasing, ϕ_p is increasing and $u'_{\lambda_n}(1) \leq 0$, the above equation yields

$$r^{N-1}\phi_p(u'_{\lambda_n}(r)) \leq \lambda_n f(v_0/2) \int_r^1 s^{N-1} ds \leq \frac{\lambda_n f(v_0/2)}{N}.$$

Using the properties of ϕ_p, facts that $f(v_0/2) < 0$ and $1/r^{N-1} > 1$ yields

$$u'_{\lambda_n}(r) \leq \phi_{p'}\left(\frac{\lambda_n f(v_0/2)}{N r^{N-1}}\right) = -\left|\frac{f(v_0/2)}{N r^{N-1}}\right|^{p'-1} \lambda_n^{p'-1} < -L\lambda_n^{\frac{1}{p-1}},$$

where $1/p + 1/p' = 1$ and $L := \left|\frac{f(v_0/2)}{N}\right|^{p'-1} > 0$. This gives $-u'_{\lambda_n}(r) > L\lambda_n^{\frac{1}{p-1}}$ and hence for $r \in [3/4, 1)$, we have

$$u_{\lambda_n}(r) = -\int_r^1 u'_{\lambda_n}(s)ds > L\lambda_n^{\frac{1}{p-1}} \int_r^1 ds = L\lambda_n^{\frac{1}{p-1}}(1 - r).$$

In particular, for $r = 4/5 \in [3/4, 1)$, we have

$$u_{\lambda_n}(4/5) \geq L\lambda_n^{\frac{1}{p-1}}/5.$$

Taking λ_n large enough, say for $\lambda_n \geq \left(\frac{5u_0}{2L}\right)^{p-1}$, we arrive at the contradiction $u_{\lambda_n}\left(\frac{4}{5}\right) \geq \frac{u_0}{2}$.

Case 2: $u_{\lambda_n}(r) \neq \frac{u_0}{c}$ and $v_{\lambda_n}(r) = \frac{v_0}{\tilde{c}}$ for all $r \in \left[\frac{3}{4}, 1\right)$. Proceeding as in Case 1 and observing that $v_{\lambda_n}(r) = \frac{v_0}{\tilde{c}} < \frac{v_0}{2}$, we arrive at the same contradiction.

This proves the lemma. ■

By the mean value theorem, there exist $r_3 \in (r_1, r_2)$ and $\widetilde{r}_3 \in (\widetilde{r}_1, \widetilde{r}_2)$ such that

$$|u'(r_3)| = \left|\frac{u(r_2) - u(r_1)}{r_2 - r_1}\right| \leq \frac{\frac{U_*}{2}}{\frac{1}{4}} = 2U_* \leq 2U_0$$

and

$$|v'(\widetilde{r}_3)| = \left|\frac{v(\widetilde{r}_2) - v(\widetilde{r}_1)}{\widetilde{r}_2 - \widetilde{r}_1}\right| \leq \frac{\frac{V_*}{2}}{\frac{1}{4}} = 2V_* \leq 2V_0.$$

Now we show that u' and v' are bounded near the boundary, that is for r close to 1.

Lemma 3.4. *There exist positive constants K_1 and K_2 (both independent of λ) such that*

$$|u'(r)| \leq K_1 \quad \textit{for all} \quad r \in [r_3, 1) \quad \textit{and} \quad |v'(r)| \leq K_2 \quad \textit{for all} \quad r \in [\tilde{r}_3, 1).$$

Proof: Let r_f, r_g be such that $u(r_g) = u_0, v(r_f) = v_0$ where u_0, v_0 are the unique zeros of g and f respectively. We first claim (a) $r_3 \in [r_f,\, 1)$ and (b) $\widetilde{r}_3 \in [r_g,\, 1)$.

We will establish (a) and the proof of (b) follows similarly.

If $r_f \le r_1$, then we are done since $r_3 > r_1$. Suppose $r_f > r_1$ and $r_1 < r_3 < r_f$. Then $|u'(r_3)| \le 2U_0$ while it follows from Lemma 3.2 that $|u'(r_1)| \to \infty$ as $\lambda \to \infty$. This is a contradiction since $f(v(r)) > 0$ for all $r \in (r_1, r_f)$ implies that u' is decreasing on (r_1, r_f) and thus $|u'(r_1)| \le |u'(r_3)|$. Hence (a) holds and similarly (b) also holds true.

Now, since $f(v(r)) < 0$ for $r \in [r_f,\, 1]$, u' is increasing on $[r_f,\, 1]$ and thus $|u'(r)| \le |u'(r_3)| \le 2U_0 =: K_1$ for all $r \in [r_f, 1]$ and hence for all $r \in [r_3, 1]$.

Similarly, we can establish that $|v'(r)| \le |v'(\widetilde{r}_3)| \le 2V_0 =: K_2$ for all $r \in [r_g, 1]$ and hence for all $r \in [\widetilde{r}_3, 1]$. This completes the proof of the lemma. ■

3.2 Proof of Theorem 1.1

We prove by contradiction. Suppose (u, v) is a positive solution of (3.1) for $\lambda > \max\{\lambda^*, \lambda^{**}\}$. Define a functional $E : [0, 1] \to \mathbb{R}$ by

$$E(r) := -\int_r^1 (\phi_p(u'(s)))'v'(s)ds - \int_r^1 (\phi_p(v'(s)))'u'(s)ds + \lambda F(v(r)) + \lambda G(u(r)). \quad (3.9)$$

Since $F(0) = 0 = G(0)$, it follows from the boundary condition $u(1) = 0 = v(1)$ that $E(1) = 0$ as the first two integrals are trivially zero when $r = 1$. It is easy to see that $E \in C^1(0, 1) \cap C[0, 1]$ and that

$$E'(r) = (\phi_p(u'(r)))'v'(r) + (\phi_p(v'(r)))'u'(r) + \lambda f(v(r))v'(r) + \lambda g(u(r))u'(r)\,.$$

First, we will analyze $E'(r)$ to determine the sign of $E(r)$ on $[0,1]$. To do so, observe that (3.1) can be rewritten as

$$\begin{cases} -(\phi_p(u'(r)))' - \frac{N-1}{r}\phi_p(u'(r)) = \lambda f(v(r)) \text{ for } \quad 0<r<1; \\ -(\phi_p(v'(r)))' - \frac{N-1}{r}\phi_p(v'(r)) = \lambda g(u(r)) \text{ for } \quad 0<r<1; \\ u'(0)=u(1)=0; \\ v'(0)=v(1)=0. \end{cases} \tag{3.10}$$

Then using (3.10) and the facts that $u'<0$, $v'<0$ and $\phi_p(\cdot)$ is an odd homeomorphism, we obtain

$$\begin{aligned} E'(r) &= (\phi_p(u'(r)))'v'(r) + (\phi_p(v'(r)))'u'(r) + \lambda f(v(r))v'(r) + \lambda g(u(r))u'(r) \\ &= (\phi_p(u'(r)))'v'(r) + (\phi_p(v'(r)))'u'(r) - (\phi_p(u'(r)))'v'(r) - \frac{N-1}{r}\phi_p(u'(r))v'(r) \\ &\quad -(\phi_p(v'(r)))'u'(r) - \frac{N-1}{r}\phi_p(v'(r))u'(r) \\ &= -\frac{N-1}{r}\phi_p(u'(r))v'(r) - \frac{N-1}{r}\phi_p(v'(r))u'(r) \\ &< 0. \end{aligned}$$

This, combined with the fact that $E(1)=0$ imply that

$$E(r) \geq 0 \quad \text{for} \quad r \in [0,1]. \tag{3.11}$$

Define $r^* := \max\{r_3, \tilde{r_3}\}$. Since $u'(r)$, $v'(r)<0$ for $r \in (0,1]$, $E(r^*)$ can be expressed as

$$E(r^*) = \int_{r^*}^{1} (\phi_p(u'(s)))'|v'(s)|ds + \int_{r^*}^{1} (\phi_p(v'(s)))'|u'(s)|ds + \lambda F(v(r^*)) + \lambda G(u(r^*)).$$

We will analyze $E(r^*)$ below to arrive at a contradiction. Since $r^* = \max\{r_3, \tilde{r_3}\}$, by Lemma 3.4,

$$\int_{r^*}^{1} (\phi_p(u'(s)))'|v'(s)|ds + \int_{r^*}^{1} (\phi_p(v'(s)))'|u'(s)|ds$$

is bounded since $|u'(r)| \leq K_1$ and $|v'(r)| \leq K_2$ for all $r \in [r^*, 1]$. Further, since $u(r_3) > u(r_2) = u_0/c \neq 0$ and $v(\widetilde{r_3}) > v(\widetilde{r_2}) = v_0/\widetilde{c} \neq 0$ for fixed $c, \widetilde{c} > 2$ and, $r_3 < r_2$ and $\widetilde{r_3} < \widetilde{r_2}$, we can see that $r^* \not\to 1$ since $r_3, \widetilde{r_3} \not\to 1$. On the other hand, it is easy to see that $u(r^*) \not\to U_0$ and $v(r^*) \not\to V_0$ as $\lambda \to \infty$. Indeed, for $\lambda > \max\{\lambda^*, \lambda^{**}\}$, $u(r^*) \leq u(r_3) < u(r_1) = \frac{U_* + u_0}{2} < U_0$ and $v(r^*) \leq v(\widetilde{r_3}) < u(\widetilde{r_1}) = \frac{V_* + v_0}{2} < V_0$. Hence $F(v(r^*)) < 0$ and $G(u(r^*)) < 0$ and bounded away from zero. Thus for λ sufficiently large $E(r^*) < 0$, a contradiction to (3.11). Therefore, there is no positive radially symmetric and radially decreasing solution of (3.1) (and hence of (1.1)) for λ large. This completes the proof of the theorem.

CHAPTER IV

PROOFS OF THEOREM 1.2 AND THEOREM 1.3

As discussed in Section 2.2, studying solutions of (1.3) can be reduced to the study of positive solutions of the singular system:

$$\begin{cases} -u''(s) = \lambda h_1(s) f(v(s)), \quad 0 < s < 1; \\ -v''(s) = \lambda h_2(s) g(u(s)), \quad 0 < s < 1; \\ \quad u(0) = u(1) = 0, \quad v(0) = v(1) = 0, \end{cases} \tag{4.1}$$

where $h_i(s) = \frac{r_0^2}{(N-2)^2} s^{\frac{-2(N-1)}{(N-2)}} K_i(r_0 s^{\frac{1}{2-N}}), i = 1, 2$. We note that the assumption (A_3) implies that $\lim_{s \to 0} h_i(s) = \infty$, for $i = 1, 2$, $\hat{h} := \inf_{t \in (0,1)} \{h_1(t), h_2(t)\} > 0$, and there exist $d > 0$, $\beta \in (0, 1)$ such that

$$h_i(s) \leq \frac{d}{s^\beta} \text{ for } s \in (0, 1], \quad \text{and for} \quad \text{i} = 1, 2. \tag{4.2}$$

4.1 Auxiliary Problem

We first establish some useful results for solutions to the following auxiliary system:

$$\begin{cases} -u''(s) &= b_1 h_1(s) |v(s) + l|^{q_1}, \quad 0 < s < 1; \\ -v''(s) &= b_2 h_2(s) |u(s) + l|^{q_2}, \quad 0 < s < 1; \\ u(0) &= u(1) = 0, \quad v(0) = v(1) = 0, \end{cases} \tag{4.3}$$

where $l \geq 0$ is a parameter. $\Big($Clearly, any solution (u_l, v_l) of (4.3) for $l > 0$ must satisfy $u_l(s) > 0, v_l(s) > 0$ for $s \in (0,1)$. This is also true for any nontrivial solution when $l = 0\Big)$. Now we prove:

Lemma 4.1. (i) *There exists $l_0 > 0$ such that* (4.3) *has no solution if $l \geq l_0$.*

(ii) *For each $l \in [0, l_0)$, there exists $M > 0$ (independent of l) such that if (u_l, v_l) is a solution of* (4.3)*, then* $\max\{||u_l||_\infty, ||v_l||_\infty\} \leq M$.

Proof: (i): Let $\lambda_1 := \pi^2$, $\phi_1 := sin(\pi s)$. Here λ_1 is the principal eigenvalue and ϕ_1 the corresponding eigenfunction of $-\phi''(s) = \lambda\phi(s)$ in $(0,1)$ with $\phi(0) = 0 = \phi(1)$. Let $a > \frac{\lambda_1}{b_1 b_2 \hat{h}}, c > 0$ be such that $(s+l)^{q_i} \geq as - c$ for all $s \geq 0$ and for $i = 1, 2$. Now let (u_l, v_l) be a solution of (4.3). Multiplying (4.3) by ϕ_1 and integrating, we obtain

$$\lambda_1 \int_0^1 u_l \phi_1 ds = b_1 \int_0^1 h_1(s)(v_l + l)^{q_1} \phi_1 ds \geq b_1 \int_0^1 h_1(s)(av_l - c)\phi_1 ds,$$

and

$$\lambda_1 \int_0^1 v_l \phi_1 ds = b_2 \int_0^1 h_2(s)(u_l + l)^{q_2} \phi_1 ds \geq b_2 \int_0^1 h_2(s)(au_l - c)\phi_1 ds.$$

Then from the above inequalities we obtain

$$\int_0^1 v_l \phi_1 ds \leq \frac{1}{ab_1\hat{h}} \left(\lambda_1 \int_0^1 u_l \phi_1 ds + b_1 c \|h_1\|_1 \right),$$

and

$$\int_0^1 u_l \phi_1 ds \leq \frac{1}{ab_2\hat{h}} \left(\lambda_1 \int_0^1 v_l \phi_1 ds + b_2 c \|h_2\|_1 \right),$$

where $\|h_i\|_1 := \int_0^1 h_i(s)ds < \infty$ for $i = 1, 2$. Hence we deduce that

$$\int_0^1 u_l \phi_1 ds \leq \frac{m_1}{m} =: m_2(\text{say}),$$

where $m := (ab_2\hat{h} - \frac{\lambda_1^2}{ab_1\hat{h}})$, and $m_1 := \frac{\lambda_1 c\|h_1\|_1}{a\hat{h}} + b_2 c\|h_2\|_1$. This implies

$$\int_0^1 (v_l + l)^{q_1} \phi_1 ds \leq \frac{\lambda_1 m_2}{b_1 \hat{h}} =: m_3(\text{say}).$$

In particular, this implies $\int_{\frac{1}{4}}^{\frac{3}{4}} l^{q_1}\, \mathrm{d}s \leq \frac{m_3}{\inf\limits_{[\frac{1}{4},\frac{3}{4}]} \phi_1}$. Since m_3 is independent of l, clearly this is a contradiction for $l >> 1$, and hence there must exists an $l_0 > 0$ such that for $l \geq l_0$, (4.3) has no solution.

(ii): Assume the contrary. Then, without loss of generality we can assume there exists $\{l_n\} \subset (0, l_0)$ such that $\|u_{l_n}\|_\infty \to \infty$ as $n \to \infty$. Clearly $u''_{l_n}(s) < 0$, and $v''_{l_n}(s) < 0$ for all $s \in (0, 1)$. Let $s_{1(l_n)} \in (0, 1)$, $s_{2(l_n)} \in (0, 1)$ be the points at which u_{l_n} and v_{l_n} attain their maximums. Now since $u''_{l_n}(s) < 0$ for all $s \in (0, 1)$, we have:

$$u_{l_n}(s) \geq \begin{cases} \frac{s u_{l_n}(s_{1(l_n)})}{s_{1(l_n)}} \text{ for } s \in (0, s_{1(l_n)}), \\ \frac{(1-s) u_{l_n}(s_{1(l_n)})}{1 - s_{1(l_n)}} \text{ for } s \in (s_{1(l_n)}, 1). \end{cases}$$

Hence $u_{l_n}(s) \geq \min\left\{\frac{s\|u_{l_n}\|_\infty}{s_{1(l_n)}}, \frac{(1-s)\|u_{l_n}\|_\infty}{1 - s_{1(l_n)}}\right\}$, and in particular, for $s \in [\frac{1}{4}, \frac{3}{4}]$,

$$u_{l_n}(s) \geq \min\left\{\frac{1}{4}\|u_{l_n}\|_\infty, \frac{1}{4}\|u_{l_n}\|_\infty\right\} = \frac{1}{4}\|u_{l_n}\|_\infty.$$

Let $\tilde{s_{l_n}}, \bar{s_{l_n}} \in [\frac{1}{4}, \frac{3}{4}]$ be such that $\min\limits_{[\frac{1}{4},\frac{3}{4}]} u_{l_n}(s) = u_{l_n}(\tilde{s_{l_n}})$, and $\min\limits_{[\frac{1}{4},\frac{3}{4}]} v_{l_n}(s) = v_{l_n}(\bar{s_{l_n}})$.

Now for $s \in [\frac{1}{4}, \frac{3}{4}]$,

$$v_{l_n}(s) \geq b_2 \hat{h} \tilde{m} \int_{\frac{1}{4}}^{\frac{3}{4}} |u_{l_n}(t) + l|^{q_2} dt,$$

where $\tilde{m} := \min_{[\frac{1}{4},\frac{3}{4}]\times[\frac{1}{4},\frac{3}{4}]} G(s,t)(> 0)$, and G is the Green's function of $-Z''$ with $Z(0) = 0 = Z(1)$. In particular, $v_{l_n}(\bar{s_{l_n}}) \geq b_2 \hat{h} \frac{\tilde{m}}{2} (u_{l_n}(\tilde{s_{l_n}}))^{q_2}$. Similarly $u_{l_n}(\tilde{s_{l_n}}) \geq b_1 \hat{h} \frac{m}{2} (v_{l_n}(\bar{s_{l_n}}))^{q_1}$. Hence, there exists a constant $A > 0$ such that

$$u_{l_n}(\tilde{s_{l_n}}) \geq A\Big(u_{l_n}(\tilde{s_{l_n}})\Big)^{q_1 q_2}.$$

This is a contradiction since $q_1 q_2 > 1$ and $u_{l_n}(\tilde{s_{l_n}}) \geq \frac{1}{4}\|u_{l_n}\|_\infty \to \infty$ as $n \to \infty$. Thus (ii) holds. $\blacksquare$

4.2 Proof of Theorem 1.2

We first extend f and g as even functions on $\mathbb{R}$ by setting $f(-s) = f(s)$ and $g(-s) = g(s)$. Then we use the rescaling, $\lambda = \gamma^\delta, w_1 = \gamma u$, and $w_2 = \gamma^\theta v$ with $\gamma > 0$, $\theta = \frac{q_2+1}{q_1+1}$, and $\delta = \frac{q_1 q_2 - 1}{q_1+1}$. With this rescaling, (4.1) reduces to

$$\begin{cases} -w_1''(s) = F(s,\gamma,w_2), \quad 0 < s < 1; \\ -w_2''(s) = G(s,\gamma,w_1), \quad 0 < s < 1; \\ w_1(0) = w_1(1) = 0, \quad w_2(0) = w_2(1) = 0, \end{cases} \tag{4.4}$$

where

$$\begin{aligned} F(s,\gamma,w_2) &:= \gamma^{1+\delta} h_1(s)\Big(f(\tfrac{w_2}{\gamma^\theta}) - b_1|\tfrac{w_2}{\gamma^\theta}|^{q_1}\Big) + b_1|w_2|^{q_1} h_1(s), \text{ and} \\ G(s,\gamma,w_1) &:= \gamma^{\theta+\delta} h_2(s)\Big(g(\tfrac{w_1}{\gamma}) - b_2|\tfrac{w_1}{\gamma}|^{q_2}\Big) + b_2|w_1|^{q_2} h_2(s)\,. \end{aligned}$$

Note that by (A_2), $F(s,\gamma,w_2) \to b_1|w_2|^{q_1}h_1(s)$ and $G(s,\gamma,w_1) \to b_2|w_1|^{q_2}h_2(s)$ as $\gamma \to 0$. Hence we can continuously extend $F(s,\gamma,w_2)$ and $G(s,\gamma,w_1)$ to $F(s,0,w_2) = b_1|w_2|^{q_1}h_1(s)$ and $G(s,0,w_1) = b_2|w_1|^{q_2}h_2(s)$, respectively. Note that proving (4.1) has a positive solution for λ small is equivalent to proving (4.4) has a solution (w_1, w_2) with $w_1 > 0, w_2 > 0$ in $(0,1)$ for small $\gamma > 0$. We will achieve this by establishing that the limiting equation (when $\gamma = 0$)

$$\begin{cases} -w_1''(s) = F(s,0,w_2) = b_1h_1(s)|w_2|^{q_1}, \quad 0 < s < 1; \\ -w_2''(s) = G(s,0,w_1) = b_2h_2(s)|w_1|^{q_2}, \quad 0 < s < 1; \\ \qquad w_1(0) = w_1(1) = 0; \quad w_2(0) = w_2(1) = 0, \end{cases} \tag{4.5}$$

(which is same as (4.3) with $l = 0$) has a positive solution that persists for small $\gamma > 0$.

Let $X = C_0[0,1] \times C_0[0,1]$ be the Banach space equipped with $\|\underline{w}\|_X = \|(w_1,w_2)\|_X := \max\{\|w_1\|_\infty, \|w_2\|_\infty\}$, where $\|.\|_\infty$ denotes the usual supremum norm in $C_0([0,1])$. Then for fixed $\gamma \geq 0$, we define the map $S(\gamma,.) : X \to X$ by

$$S(\gamma,\underline{w}) := \underline{w} - \Big(K(F(s,\gamma,w_2)), K(G(s,\gamma,w_1))\Big)$$

where $K(H(s,\gamma,Z(s))) = \int_0^1 G(t,s)H(t,\gamma,Z(t))dt$. Note that $F(s,\gamma,.), G(s,\gamma,.) : C_0([0,1]) \to L^1(0,1)$ are continuous and $K : L^1(0,1) \to C_0^1([0,1])$ is compact. Hence $S(\gamma,\underline{w})$ is a compact perturbation of the identity. Clearly for $\gamma > 0$, if $S(\gamma,\underline{w}) = \underline{0}$, then $\underline{w} = (w_1,w_2)$ is a solution of (4.4), and if $S(0,\underline{w}) = \underline{0}$, then $\underline{w} = (w_1,w_2)$ is a solution of (4.5).

We first establish:

Lemma 4.2. *There exists $R > 0$ such that $S(0, \underline{w}) \neq \underline{0}$ for all $\underline{w} = (w_1, w_2) \in X$ with $\|\underline{w}\|_X = R$ and deg$(S(0,.), B_R(\underline{0}), \underline{0}) = 0$.*

Proof: Define $S^l(0, \underline{w}) : X \to X$ by

$$S^l(0, \underline{w}) := \underline{w} - \Big(K(b_1 h_1(s)|w_2 + l|^{q_1}), K(b_2 h_2(s)|w_1 + l|^{q_2})\Big)$$

for $l \geq 0$. (Note $S^0(0, \underline{w}) = S(0, \underline{w})$). By Lemma 4.1, if $l \geq l_0$ then $S^l(0, \underline{w}) \neq \underline{0}$ and if $S^l(0, \underline{w}) = \underline{0}$ for $l \in [0, l_0)$, then $\|w\|_X \leq M$. This implies that there exists $R \gg 1$ such that $S^l(0, \underline{w}) \neq \underline{0}$ for $\underline{w} \in \partial B_R(\underline{0})$ for any $l \geq 0$. Also, since (4.3) has no solution for $l \geq l_0$, $\deg(S^{l_0}(0,.), B_R(\underline{0}), \underline{0}) = 0$. Hence, using the homotopy invariance of degree with the parameter $l \in [0, l_0]$ we get

$$\deg(S(0,.), B_R(\underline{0}), \underline{0}) = \deg(S^{l_0}(0,.), B_R(\underline{0}), \underline{0}) = 0,$$

which completes the proof of lemma 4.2. ■

Lemma 4.3. *There exists $r \in (0, R)$ small enough such that $S(0, \underline{w}) \neq \underline{0}$ for all $\underline{w} = (w_1, w_2) \in X$ with $\|\underline{w}\|_X = r$ and deg$(S(0,.), B_r(\underline{0}), \underline{0}) = 1$.*

Proof: Define $T^\tau(0, \underline{w}) : X \to X$ by

$$T^\tau(0, \underline{w}) := \underline{w} - \Big(K(\tau b_1 h_1(s)|w_2|^{q_1}), K(\tau b_2 h_2(s)|w_1|^{q_2})\Big)$$

for $\tau \in [0, 1]$. Clearly $T^1(0, \underline{w}) = S(0, \underline{w})$, and $T^0(0, \underline{w}) = I$ is the identity operator.

Note that $T^\tau(0, \underline{w}) = \underline{0}$ if $\underline{w} = (w_1, w_2)$ is a solution of

$$\begin{cases} -w_1''(s) = \tau b_1 h_1(s)|w_2|^{q_1}, \quad 0 < s < 1; \\ -w_2''(s) = \tau b_2 h_2(s)|w_1|^{q_2}, \quad 0 < s < 1; \\ w_1(0) = w_1(1) = 0; \quad w_2(0) = w_2(1) = 0, \end{cases} \tag{4.6}$$

and for $\tau = 1$, (4.6) coincides with (4.5). Assume to the contrary that (4.6) has a solution $\underline{w} = (w_1, w_2)$ with $\|\underline{w}\|_X = \tilde{r} > 0$. Without loss of generality assume $\|w_1\|_\infty = \tilde{r}$. Now,

$$w_1(s) = \tau \int_0^1 G(s,t) b_1 h_1(s) |w_2|^{q_1} ds.$$

Then $\|w_1\|_\infty \le \tilde{C} \|w_2\|_\infty^{q_1}$ for some constant $\tilde{C} > 0$ independent of $\tau \in [0, 1]$. Similarly $\|w_2\|_\infty \le \hat{C} \|w_1\|_\infty^{q_2}$ for some constant $\hat{C} > 0$. This implies that

$$\tilde{r} = \|w_1\|_\infty \le C \|w_1\|_\infty^{q_1 q_2} = C \tilde{r}^{q_1 q_2}$$

for some constant $C > 0$. But $q_1 q_2 > 1$, and hence this is a contradiction if $\tilde{r} > 0$ is small. Thus there exists small $r > 0$ such that (4.6) has no solution $\underline{w}$ with $\|\underline{w}\|_X = r$ for all $\tau \in [0, 1]$.

Now using the homotopy invariance of degree with the parameter $\tau \in [0, 1]$, in particular using the values $\tau = 1$ and $\tau = 0$, we obtain

$$\deg(S(0, .), B_r(\underline{0}), \underline{0}) = \deg(T^1(0, .), B_r(\underline{0}), \underline{0}) = \deg(T^0(0, .), B_r(\underline{0}), \underline{0}) = 1\,.$$

By Lemma 4.2 and Lemma 4.3, with $0 < r < R$, we conclude that

$$\deg(S(0,.), B_R(\underline{0}) \setminus \overline{B_r}(\underline{0}), \underline{0}) = -1,$$

and hence (4.5) has a solution $\underline{w} = (w_1, w_2)$ with $w_1 > 0, w_2 > 0$ in $(0,1)$, and $r < \|w\|_X < R$. Now we show that the solution obtained above (when $\gamma = 0$) persists for small $\gamma > 0$ and remains positive componentwise. ■

Lemma 4.4. *Let R, r be as in Lemmas* 4.2, 4.3*, respectively. Then there exists $\gamma_0 > 0$ such that*

(a) *deg$(S(\gamma,.), B_R(\underline{0}) \setminus \overline{B_r}(\underline{0}), \underline{0}) = -1$ for all $\gamma \in [0, \gamma_0]$.*

(b) *If $S(\gamma, \underline{w}) = \underline{0}$ for $\gamma \in [0, \gamma_0]$ with $r < \|\underline{w}\|_X < R$, then $w_1 > 0, w_2 > 0$ in $(0,1)$.*

Proof: (a): We first show that there exists $\gamma_0 > 0$ such that $S(\gamma, \underline{w}) \neq \underline{0}$ for all $\underline{w} = (w_1, w_2) \in X$ with $\|\underline{w}\|_X \in \{R, r\}$, for all $\gamma \in [0, \gamma_0]$. Suppose to the contrary that there exists $\{\gamma_n\}$ with $\gamma_n \to 0$, $S(\gamma_n, \underline{w_n}) = \underline{0}$ and $\|\underline{w_n}\|_X \in \{r, R\}$. Since $\underline{K} = (K, K) : L^1(0,1) \times L^1(0,1) \to C_0^1([0,1]) \times C_0^1([0,1])$ is compact, and $\{F(s, \gamma_n, w_{2n}), G(s, \gamma_n, w_{1n})\}$ are bounded in $L^1(0,1) \times L^1(0,1)$, $\underline{w}_n \to \underline{Z} = (Z_1, Z_2) \in C_0^1([0,1]) \times C_0^1([0,1])$ (upto a subsequence) with $\|\underline{Z}\|_X = R$ or r and $S(0, \underline{Z}) = \underline{0}$. This is a contradiction to Lemmas 4.2 or 4.3 and hence there exists $\gamma_0 > 0$ small satisfying the assertions. Now, by the homotopy invariance of degree with respect to $\gamma \in [0, \gamma_0]$,

$$\deg(S(\gamma,.), B_R(\underline{0}) \backslash \overline{B_r}(\underline{0}), \underline{0}) = \deg(S(0,.), B_R(\underline{0}) \backslash \overline{B_r}(\underline{0}), \underline{0}) = -1 \text{ for all } \gamma \in [0, \gamma_0].$$

(b): Assume to the contrary that there exists $\gamma_n \to 0$ and a corresponding solution $\underline{w}_n = (w_{1n}, w_{2n})$ such that $r < \|\underline{w}_n\|_X < R$ and

$$\Omega_n := \{x \in (0,1) | w_{1n}(x) \leq 0 \text{ or } w_{2n}(x) \leq 0\} \neq \emptyset \, .$$

Arguing as before, $\underline{w_n} \to \underline{Z} \in C_0^1([0,1]) \times C_0^1([0,1])$ with $S(0, \underline{Z}) = \underline{0}$ (upto a subsequence). Note that $\underline{Z} \not\equiv \underline{0}$ since $\|\underline{Z}\|_X \geq r > 0$. By the strong maximum principle $Z_1 > 0, Z_2 > 0, Z_1'(0) > 0, Z_2'(0) > 0, Z_1'(1) < 0$ and $Z_2'(1) < 0$. Now suppose there exists $\{x_n\} \in (0,1)$ with $\{x_n\} \in \Omega_n$ and $w_{1n}(x_n) \leq 0$. Then $\{x_n\}$ must have a subsequence (renamed as $\{x_n\}$ itself) such that $x_n \to \tilde{x} \in [0,1]$. But $Z_1 > 0$ in $(0,1)$ implies that $\tilde{x} \in \{0,1\}$. Suppose $\tilde{x} = 0$. Since $w_{1n}(x_n) \leq 0$ and $w_{1n}(0) = 0$, there exists $y_n \in (0, x_n)$ such that $w_{1n}'(y_n) \leq 0$ and hence taking the limit as $n \to \infty$ we will have $Z_1'(0) \leq 0$, which is a contradiction since $Z_1'(0) > 0$. Similar contradiction follows if $\tilde{x} = 1$, using the fact that $Z_1'(1) < 0$. Further, contradictions can be achieved if there exists $\{x_n\} \in \Omega$ with $\{x_n\} \in \Omega_n$ and $w_{2n}(x_n) \leq 0$ using the facts that $Z_2'(0) > 0$ and $Z_2'(1) < 0$. This completes the proof of the lemma. ∎

Now we can easily deduce Theorem 1.2. From Lemma 4.4, since $\underline{w} = (w_1, w_2)$ is a positive solution of (4.4) for γ small, $(u, v) = (\gamma^{-1} w_1, \gamma^{-\theta} w_2)$ with $\theta = \frac{q_2+1}{q_1+1}$ is a positive solution of (4.1) for $\lambda = \gamma^\delta$ where $\delta = \frac{q_1 q_2 - 1}{q_1+1}$. Further, since $w_1 > 0$ and $w_2 > 0$ in $(0,1)$ for $\gamma \in [0, \gamma_0]$, $\|u\|_\infty \to \infty$ and $\|v\|_\infty \to \infty$ as $\lambda (= \gamma^\delta) \to 0$. This completes the proof of Theorem 1.2.

4.3 Proof of Theorem 1.3

We will prove Theorem 1.3 by studying the singular system (4.1). We first recall from [CSS12] that when (A_5) is satisfied, h_1 is decreasing in $(0,1]$, and via energy analysis one can prove that nonnegative solution of (4.1) must be positive in $(0,1)$, have a unique interior maximum with maximum value greater than θ, where θ is the unique positive zero of $\tilde{F}(s) = \int_0^s \tilde{f}(y)dy$. Further, if the solution u exists for $\lambda \gg 1$, and $s_1, \hat{s_1} \in (0,1)$ are such that $\hat{s_1} > s_1$, $u(s_1) = u(s_2) = \beta$, where $\beta > 0$ is the unique zero of $\tilde{f}$, then there exists a constant C such that $s_1 \leq C\lambda^{-\frac{1}{2}}$ and $(1 - s_1) \leq C\lambda^{-\frac{1}{2}}$. Hence we can assume $(\hat{s_1} - s_1) > \frac{1}{2}$ for $\lambda \gg 1$. Now we provide the proof of Theorem 1.3. Let $v := u - \beta$. Then $v > 0$ in $(s_1, \hat{s_1})$ and satisfies:

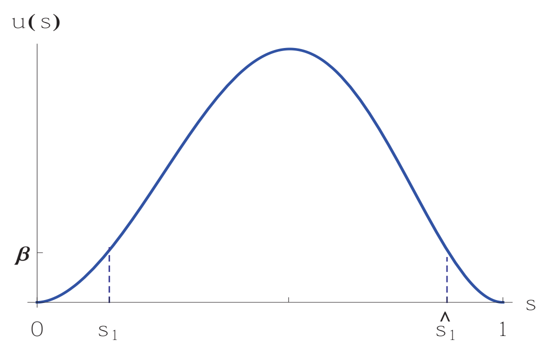

Figure 5. Graph of u

$$\begin{cases} -v'' = \lambda h_1(s) \dfrac{\tilde{f}(u)}{u-\beta} v, \quad s_1 < s < \hat{s_1} \\ v(s_1) = v(\hat{s_1}) = 0. \end{cases}$$

Note that $\phi(s) = -\left(\sin\left(\frac{\pi(s-s_1)}{(\hat{s_1}-s_1)}\right)\right) > 0$ in $(s_1, \hat{s_1})$, $\phi(s_1) = \phi(\hat{s_1}) = 0$, and satisfies $-\phi'' = \frac{\pi^2}{(\hat{s_1}-s_1)^2}\phi$ in $(s_1, \hat{s_1})$. Hence using the fact that $\int_{s_1}^{\hat{s_1}} (-\phi v'' + v\phi'')ds = 0$, we obtain $\int_{s_1}^{\hat{s_1}} \left(\lambda \frac{\tilde{f}(u)}{u-\beta} h_1(s) - \frac{\pi^2}{(\hat{s_1}-s_1)^2}\right) v\phi ds = 0$. In particular,

$$\lambda \frac{\tilde{f}(u)}{u-\beta} h_1(s) = \frac{\pi^2}{(\hat{s_1}-s_1)^2}, \text{ for some } s_\lambda \in (s_1, \hat{s_1}). \tag{4.7}$$

But $\hat{h} = \inf\limits_{(0,1)} h_1(s) > 0$, and $(\hat{s_1} - s_1) > \frac{1}{2}$ for $\lambda \gg 1$. Thus clearly (4.7) can hold when $\lambda \to \infty$, only if $Z = u(s_\lambda) \to \infty$ with $\frac{\tilde{f}(u(s_\lambda))}{u(s_\lambda)-\beta} \to 0$. But by (A_4), this is not possible since $\lim\limits_{Z\to\infty} \frac{\tilde{f}(Z)}{Z} \geq m_0 > 0$. Hence the nonnegative solution does not exists for λ large.

CHAPTER V

PROOFS OF THEOREMS 1.4 - 1.6

5.1 Proof of Theorem 1.4

First we discuss certain inequalities that will be crucial in the construction of a subsolution. Here $p, q > 1$ and $\alpha_i \in [0, 1)$, $i = 1, 2$ are as in Theorem 1.4. For $\theta \in \mathbb{R}$, define

$$P(\theta) := p\theta^2 + [p(q-1) - \alpha_1 q]\theta - \alpha_1 q(p-1).$$

Then $P(\theta)$ has two distinct real roots

$$\theta_{1,P} := \frac{\alpha_1 q - p(q-1) - \sqrt{[\alpha_1 q - p(q-1)]^2 + 4pq\alpha_1(p-1)}}{2p} < 0,$$

and

$$\theta_{2,P} := \frac{\alpha_1 q - p(q-1) + \sqrt{[\alpha_1 q - p(q-1)]^2 + 4pq\alpha_1(p-1)}}{2p} > 0\,.$$

However, $P(1) = p + p(q-1) - \alpha_1 q - \alpha_1 q(p-1) = pq(1-\alpha_1) > 0$ and hence $0 < \theta_{2,P} < 1$, see Figure 6 below. Similarly,

$$Q(\theta) := q\theta^2 + [q(p-1) - \alpha_2 p]\theta - \alpha_2 p(q-1)$$

has two distinct real roots $\theta_{1,Q} < 0 < \theta_{2,Q} < 1$.

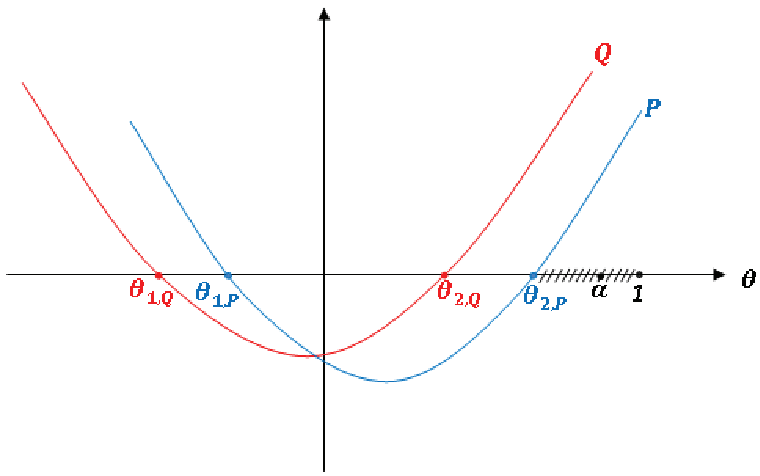

Figure 6. Graph of $P(\theta)$ and $Q(\theta)$

Let

$$\alpha \in [\max\{\theta_{2,P}, \theta_{2,Q}\}, 1)\,.$$

Then since $P(\alpha) > 0$ and $Q(\alpha) > 0$, we have

$$\begin{cases} \frac{\alpha p}{p-1+\alpha} > \frac{\alpha_1 q}{q-1+\alpha}\,, \\ \frac{\alpha q}{q-1+\alpha} > \frac{\alpha_2 p}{p-1+\alpha}\,. \end{cases} \tag{5.1}$$

Now let ν_m be the principal eigenvalue of

$$\begin{cases} -\Delta_m \phi = \nu |\phi|^{m-2}\phi \quad \text{in} \quad \Omega\,; \\ \phi = 0 \quad \text{in} \quad \partial\Omega\,. \end{cases} \tag{5.2}$$

Then the corresponding eigenfunction, $\phi_m \in C^1(\overline{\Omega})$, is of one sign in Ω and $\frac{\partial \phi_m}{\partial \eta} < 0$ on $\partial\Omega$. Without loss of generality, we normalize ϕ_m so that $\phi_m > 0$ in Ω and $\|\phi_m\|_\infty = 1$.

Furthermore, since $|\nabla\phi_m| \neq 0$ near $\partial\Omega$, and $\phi_m > 0$ in Ω, there exist $\delta, a > 0$ and $0 < \sigma < 1$ such that for $m = p, q$

$$\begin{cases} \nu_m\phi_m^m - \frac{(m-1)(1-\alpha)}{m-1+\alpha}|\nabla\phi_m|^m \leq -a & \text{on} \quad \overline{\Omega}_\delta; \\ \phi_m \geq \sigma & \text{on} \quad \Omega\setminus\overline{\Omega}_\delta, \end{cases} \tag{5.3}$$

where $\Omega_\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\}$. Moreover, there exist domain constants $C_1 := C_1(\Omega)$, $C_2 := C_2(\Omega) > 0$ such that $C_1\phi_p \leq \phi_q$ and $C_2\phi_q \leq \phi_p$ in Ω. Let

$$(\underline{u}, \underline{v}) := \left((\frac{\lambda_1+\mu_1}{a}K_0)^{\frac{1}{p-1}}\ \frac{p-1+\alpha}{p}\ \phi_p^{\frac{p}{p-1+\alpha}}, (\frac{\lambda_2+\mu_2}{a}\overline{K}_0)^{\frac{1}{q-1}}\ \frac{q-1+\alpha}{q}\ \phi_q^{\frac{q}{q-1+\alpha}} \right)$$

where K_0 and $\overline{K}_0$ are positive constants defined by $-K_0 := \min_{s\geq 0}\{f_1(s), g_1(s)\}$ and $-\overline{K}_0 := \min_{s\geq 0}\{f_2(s), g_2(s)\}$. Observe that for $z_m := A\ \frac{m-1+\alpha}{m}\ \phi_m^{\frac{m}{m-1+\alpha}}$, we have $\nabla z_m = A\phi_m^{\frac{1-\alpha}{m-1+\alpha}}\nabla\phi_m$. Therefore, using the weak formulation of (5.2), for all $\xi \in W$, we get

$$\begin{aligned}
&\int_\Omega |\nabla z_m|^{m-2}\nabla z_m\cdot\nabla\xi\,\mathrm{d}x \\
= \;& A^{m-1}\int_\Omega \phi_m^{\frac{(1-\alpha)(m-1)}{m-1+\alpha}}|\nabla\phi_m|^{m-2}\nabla\phi_m\cdot\nabla\xi\,\mathrm{d}x \\
= \;& A^{m-1}\int_\Omega |\nabla\phi_m|^{m-2}\nabla\phi_m\cdot\left[\nabla(\xi\phi_m^{\frac{(1-\alpha)(m-1)}{m-1+\alpha}}) - \xi\frac{(1-\alpha)(m-1)}{m-1+\alpha}\ \phi_m^{\frac{-\alpha m}{m-1+\alpha}}\right]\mathrm{d}x \\
= \;& A^{m-1}\int_\Omega \Big[|\nabla\phi_m|^{m-2}\nabla\phi_m\cdot\nabla(\xi\phi_m^{\frac{(1-\alpha)(m-1)}{m-1+\alpha}}) - \\
& \frac{(1-\alpha)(m-1)}{m-1+\alpha}|\nabla\phi_m|^{m-2}\phi_m^{\frac{-\alpha m}{m-1+\alpha}}\xi\nabla\phi_m\Big]\,\mathrm{d}x \\
= \;& A^{m-1}\int_\Omega \Big[\nu_m\phi_m^{m-1}(\phi_m^{\frac{(1-\alpha)(m-1)}{m-1+\alpha}}\xi) - \\
& \frac{(1-\alpha)(m-1)}{m-1+\alpha}|\nabla\phi_m|^{m-2}\phi_m^{\frac{-\alpha m}{m-1+\alpha}}\xi\nabla\phi_m\Big]\,\mathrm{d}x \\
= \;& A^{m-1}\int_\Omega \phi_m^{\frac{-\alpha m}{m-1+\alpha}}\left[\nu_m\phi_m^m - \frac{(1-\alpha)(m-1)}{m-1+\alpha}\ |\nabla\phi_m|^m\right]\xi\,\mathrm{d}x.
\end{aligned}$$

Then, for all $\xi \in W$, $(\underline{u}, \underline{v})$ satisfies

$$\int_\Omega |\nabla \underline{u}|^{p-2}\nabla \underline{u}\cdot\nabla\xi \,\mathrm{d}x = K_0\frac{\lambda_1+\mu_1}{a}\int_\Omega \phi_p^{\frac{-\alpha p}{p-1+\alpha}}\left[\nu_p\phi_p^p - \frac{(1-\alpha)(p-1)}{p-1+\alpha}|\nabla\phi_p|^p\right]\xi\,\mathrm{d}x$$
$$\int_\Omega |\nabla \underline{v}|^{q-2}\nabla \underline{v}\cdot\nabla\xi \,\mathrm{d}x = \overline{K}_0\frac{\lambda_2+\mu_2}{a}\int_\Omega \phi_q^{\frac{-\alpha q}{q-1+\alpha}}\left[\nu_q\phi_q^q - \frac{(1-\alpha)(q-1)}{q-1+\alpha}|\nabla\phi_q|^q\right]\xi\,\mathrm{d}x\,.$$

Now on $\overline{\Omega}_\delta$, since $\|\phi_p\|_\infty = 1 = \|\phi_q\|_\infty$, using (5.1), (5.3) and the inequality $\phi_q \geq C_1\phi_p$, for large $\lambda_2+\mu_2$, we have

$$\begin{aligned}
&\int_{\Omega_\delta} |\nabla \underline{u}|^{p-2}\nabla \underline{u}\cdot\nabla\xi\,\mathrm{d}x \\
= \;& K_0\frac{\lambda_1+\mu_1}{a}\int_{\Omega_\delta}\phi_p^{\frac{-\alpha p}{p-1+\alpha}}\left[\lambda_p\phi_p^p \;-\; \frac{(1-\alpha)(p-1)}{p-1+\alpha}\;|\nabla\phi_p|^p\right]\xi\,\mathrm{d}x \\
\leq \;& -K_0(\lambda_1+\mu_1)\int_{\Omega_\delta}\phi_p^{\frac{-\alpha p}{p-1+\alpha}}\xi\,\mathrm{d}x \\
\leq \;& \lambda_1\int_{\Omega_\delta} f_1(\underline{u})\xi\,\mathrm{d}x - \frac{K_0\mu_1}{\left[(\frac{\lambda_2+\mu_2}{a}\;\overline{K}_0)^{\frac{1}{q-1}}\frac{q-1+\alpha}{q}C_1^{\frac{q}{q-1+\alpha}}\right]^{\alpha_1}}\int_{\Omega_\delta}\frac{\xi}{\phi_p^{\frac{\alpha p}{p-1+\alpha}}}\,\mathrm{d}x \\
\leq \;& \lambda_1\int_{\Omega_\delta} f_1(\underline{u})\xi\,\mathrm{d}x - \frac{K_0\mu_1}{\left[(\frac{\lambda_2+\mu_2}{a}\;\overline{K}_0)^{\frac{1}{q-1}}\frac{q-1+\alpha}{q}C_1^{\frac{q}{q-1+\alpha}}\right]^{\alpha_1}}\int_{\Omega_\delta}\frac{\xi}{\phi_p^{\frac{\alpha_1 q}{q-1+\alpha}}}\,\mathrm{d}x \\
= \;& \lambda_1\int_{\Omega_\delta} f_1(\underline{u})\xi\,\mathrm{d}x - \frac{K_0\mu_1}{\left[(\frac{\lambda_2+\mu_2}{a}\;\overline{K}_0)^{\frac{1}{q-1}}\frac{q-1+\alpha}{q}\right]^{\alpha_1}}\int_{\Omega_\delta}\frac{\xi}{(C_1\phi_p)^{\frac{\alpha_1 q}{q-1+\alpha}}}\,\mathrm{d}x \\
\leq \;& \lambda_1\int_{\Omega_\delta} f_1(\underline{u})\xi\,\mathrm{d}x - \frac{K_0\mu_1}{\left[(\frac{\lambda_2+\mu_2}{a}\;\overline{K}_0)^{\frac{1}{q-1}}\frac{q-1+\alpha}{q}\right]^{\alpha_1}}\int_{\Omega_\delta}\frac{\xi}{\phi_q^{\frac{\alpha_1 q}{q-1+\alpha}}}\,\mathrm{d}x \\
\leq \;& \lambda_1\int_{\Omega_\delta} f_1(\underline{u})\xi\,\mathrm{d}x + \mu_1\int_{\Omega_\delta}\frac{g_1(\underline{v})}{\left[(\frac{\lambda_2+\mu_2}{a}\;\overline{K}_0)^{\frac{1}{q-1}}\frac{q-1+\alpha}{q}\phi_q^{\frac{q}{q-1+\alpha}}\right]^{\alpha_1}}\xi\,\mathrm{d}x \\
= \;& \int_{\Omega_\delta}\left[\lambda_1 f_1(\underline{u}) + \mu_1\frac{g_1(\underline{v})}{\underline{v}^{\alpha_1}}\right]\xi\,\mathrm{d}x \;=\; \int_{\Omega_\delta}\left[\lambda_1 f_1(\underline{u}) + \mu_1\tilde{g}_1(\underline{v})\right]\xi\,\mathrm{d}x\,.
\end{aligned}$$

Similarly for $\lambda_1 + \mu_1$ large, it can be shown that $\underline{v}$ satisfies

$$\int_{\Omega_\delta} |\nabla \underline{v}|^{q-2} \nabla \underline{v} \cdot \nabla \xi \, \mathrm{d}x \leq \int_{\Omega_\delta} \left[\lambda_2 \widetilde{f}_2(\underline{u}) + \mu_2 g_2(\underline{v}) \right] \xi \, \mathrm{d}x \tag{5.4}$$

for all $\xi \in W$.

Next, in $\Omega \setminus \overline{\Omega}_\delta$, since $\phi_p, \phi_q \geq \sigma > 0$, by (B2), the following estimate holds for $\lambda_i + \mu_i$ sufficiently large for $i = 1, 2$

$$f_1(\underline{u}),\ \widetilde{f}_2(\underline{u}),\ \widetilde{g}_1(\underline{v}),\ g_2(\underline{v}) \geq \max\left\{ \frac{K_0}{a}\nu_p, \frac{\overline{K}_0}{a}\nu_q \right\} .$$

Therefore for $\xi \in W$, $\underline{u}$ satisfies

$$\begin{aligned}
\int_{\Omega \setminus \overline{\Omega}_\delta} |\nabla \underline{u}|^{p-2} \nabla \underline{u} \cdot \nabla \xi \, \mathrm{d}x &\leq K_0 \nu_p \frac{\lambda_1 + \mu_1}{a} \int_{\Omega \setminus \overline{\Omega}_\delta} \phi_p^{\frac{-\alpha p}{p-1+\alpha}} \ \phi_p^p \xi \, \mathrm{d}x \\
&= K_0 \nu_p \frac{\lambda_1 + \mu_1}{a} \int_{\Omega \setminus \overline{\Omega}_\delta} \phi_p^{\frac{p(p-1)}{p-1+\alpha}} \xi \, \mathrm{d}x \\
&\leq K_0 \nu_p \frac{\lambda_1 + \mu_1}{a} \int_{\Omega \setminus \overline{\Omega}_\delta} \xi \, \mathrm{d}x \\
&\leq \int_{\Omega \setminus \overline{\Omega}_\delta} [\lambda_1 f_1(\underline{u}) + \mu_1 \widetilde{g}_1(\underline{v})] \xi \, \mathrm{d}x \, .
\end{aligned}$$

Similarly, for $\xi \in W$, $\underline{v}$ satisfies

$$\int_{\Omega \setminus \overline{\Omega}_\delta} |\nabla \underline{v}|^{q-2} \nabla \underline{v} \cdot \nabla \xi \, \mathrm{d}x \leq \int_{\Omega \setminus \overline{\Omega}_\delta} [\lambda_2 \widetilde{f}_2(\underline{u}) + \mu_2 g_2(\underline{v})] \xi \, \mathrm{d}x \, .$$

Therefore, $(\underline{u}, \underline{v})$ is a subsolution of (1.5).

Now we will construct a supersolution of (1.5). For $m = p, q$, let $e_m \in C^1(\overline{\Omega})$ be the unique solution of

$$\begin{cases} -\Delta_m e = 1 \quad \text{in} \quad \Omega; \\ \quad e = 0 \quad \text{in} \quad \Omega. \end{cases} \tag{5.5}$$

It is well known that $e_m > 0$ in Ω and that $\frac{\partial e_m}{\partial \eta} < 0$ on $\partial\Omega$, where η is the outward normal on the boundary $\partial\Omega$.

Then we set

$$(\overline{u}, \overline{v}) := \left(Ce_p, [(\lambda_2 + \mu_2)\widetilde{f}_2(C\|e_p\|_\infty)]^{\frac{1}{q-1}} e_q \right).$$

For all $\xi \in W$, $\overline{u}$ satisfies

$$\int_\Omega |\nabla \overline{u}|^{p-2} \nabla \overline{u} \cdot \nabla \xi \mathrm{d}x = C^{p-1} \int_\Omega |\nabla e_p|^{p-2} \nabla e_p \cdot \nabla \xi \mathrm{d}x = C^{p-1} \int_\Omega \xi \, \mathrm{d}x. \tag{5.6}$$

Define $\overline{f}_1(s) := \max\limits_{t \in [0,s]} f_1(t)$. Then $\overline{f}_1(s)$ is nondecreasing and $\overline{f}_1(s) \geq f_1(s)$ for all $s \geq 0$. It follows from (B2), (B3) and (B4) that there exists $C > 0$ sufficiently large such that

$$C^{p-1} \geq \lambda_1 \overline{f}_1(C\|e_p\|_\infty) + \mu_1 \widetilde{g}_1([(\lambda_2 + \mu_2)\widetilde{f}_2(C\|e_p\|_\infty)]^{\frac{1}{q-1}} \|e_q\|_\infty).$$

Now since $\overline{f}_1$ and $\widetilde{g}_1$ are nondecreasing, we have

$$\begin{aligned} C^{p-1} &\geq \lambda_1 \overline{f}_1(Ce_p) + \mu_1 \widetilde{g}_1([(\lambda_2 + \mu_2)\widetilde{f}_2(C\|e_p\|_\infty)]^{\frac{1}{q-1}} e_q) \\ &= \lambda_1 \overline{f}_1(\overline{u}) + \mu_1 \widetilde{g}_1(\overline{v}) \geq \lambda_1 f_1(\overline{u}) + \mu_1 \widetilde{g}_1(\overline{v}). \end{aligned} \tag{5.7}$$

Combining (5.6) and (5.7) yields

$$\int_\Omega |\nabla \overline{u}|^{p-2} \nabla \overline{u} \cdot \nabla \xi \mathrm{d}x \geq \lambda_1 \int_\Omega f_1(\overline{u}) \xi \mathrm{d}x + \mu_1 \int_\Omega \widetilde{g}_1(\overline{v}) \xi \, \mathrm{d}x. \tag{5.8}$$

Next, it is easy to see that $\overline{v} = \left[(\lambda_2 + \mu_2) \widetilde{f}_2(C\|e_p\|_\infty) \right]^{\frac{1}{q-1}} e_q$ satisfies

$$\begin{aligned}
& \int_\Omega |\nabla \overline{v}|^{q-2} \nabla \overline{v} \cdot \nabla \xi \mathrm{d}x \\
= \ & (\lambda_2 + \mu_2) \widetilde{f}_2(C\|e_p\|_\infty) \int_\Omega |\nabla e_q|^{q-2} \nabla e_q \cdot \nabla \xi \, \mathrm{d}x \\
= \ & (\lambda_2 + \mu_2) \widetilde{f}_2(C\|e_p\|_\infty) \int_\Omega \xi \, \mathrm{d}x \\
= \ & [\lambda_2 \widetilde{f}_2(C\|e_p\|_\infty) + \mu_2 \widetilde{f}_2(C\|e_p\|_\infty)] \int_\Omega \xi \, \mathrm{d}x \,.
\end{aligned} \tag{5.9}$$

By (B2) and (B3), for $C > 0$ sufficiently large, we have

$$\widetilde{f}_2(C\|e_p\|_\infty) \geq \overline{g}_2([(\lambda_2 + \mu_2)\widetilde{f}_2(C\|e_p\|_\infty)]^{\frac{1}{q-1}} \|e_q\|_\infty) \geq \overline{g}_2([(\lambda_2 + \mu_2)\widetilde{f}_2(C\|e_p\|_\infty)]^{\frac{1}{q-1}} e_q)$$

where $\overline{g}_2(s) := \max_{t \in [0,s]} g_2(t)$ is a nondecreasing function. Then since $\overline{g}_2(s) \geq g_2(s)$ for $s \geq 0$, (5.9) yields

$$\begin{aligned}
& \int_\Omega |\nabla \overline{v}|^{q-2} \nabla \overline{v} \cdot \nabla \xi \mathrm{d}x \\
\geq \ & \lambda_2 \int_\Omega \widetilde{f}_2(Ce_p) \xi \mathrm{d}x + \mu_2 \int_\Omega g_2([(\lambda_2 + \mu_2)\widetilde{f}_2(C\|e_p\|_\infty)]^{\frac{1}{q-1}} e_q) \xi \, \mathrm{d}x \\
= \ & \lambda_2 \int_\Omega \widetilde{f}_2(\overline{u}) \xi \mathrm{d}x + \mu_2 \int_\Omega g_2(\overline{v}) \xi \, \mathrm{d}x
\end{aligned}$$

and hence $(\overline{u}, \overline{v})$ is a supersolution.

Clearly for $C >> 1$, $(\underline{u}, \underline{v}) \leq (\overline{u}, \overline{v})$ and this completes the proof.

5.2 Proof of Theorem 1.5

We will establish our result by constructing subsolutions and supersolutions satisfying the hypotheses of Proposition 2.2. Since $f_1(0) = 0 = f_2(0) = g_1(0) = g_2(0)$, obviously $(\underline{u}, \underline{v}) = (0, 0)$ is a solution of (1.5) and hence a subsolution of (1.5).

It follows from previous section, with $\alpha_1 = \alpha_2 = 0$, that

$$(\overline{u}, \overline{v}) := \left(Ce_p, [(\lambda_2 + \mu_2) f_2(C\|e_p\|_\infty)]^{\frac{1}{q-1}} e_q\right)$$

is a supersolution of (1.5) for C sufficiently large using (A2)-(A5).

To construct a strict subsolution, $(\underline{u}_s, \underline{v}_s)$, we use the following result:

Proposition 5.1. *[AS07, Theorem A] Suppose $\hat{f}_i(s), \hat{g}_i(s)$ are nondecreasing for both $i = 1, 2$ and bounded below by $-K$ for some $K > 0$. Suppose $\hat{f}_i(s), \hat{g}_i(s)$ satisfy (A4)-(A5). Then*

$$\begin{cases} -\Delta_p u = \lambda_1 \hat{f}_1(u) + \mu_1 \hat{g}_1(v) & \text{in} \quad \Omega; \\ -\Delta_q v = \lambda_2 \hat{f}_2(u) + \mu_2 \hat{g}_2(v) & \text{in} \quad \Omega; \\ u = v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{5.10}$$

has a positive solution $(\hat{u}, \hat{v})$ for $\lambda_i + \mu_i$ large for $i = 1, 2$.

Let $\hat{f}_i(s)$, $\hat{g}_i(s) : [0, \infty) \to \mathbb{R}$ be C^1 functions such that $\hat{f}_i(s) < f_i(s)$ and $\hat{g}_i(s) < g_i(s)$ for all $s \geq 0$ satisfying the hypotheses of Proposition 5.1. Let $(\hat{u}, \hat{v})$ be a positive solution of (5.10) for $\lambda_i + \mu_i$ large for $i = 1, 2$. Then clearly $(\underline{u}_s, \underline{v}_s) = (\hat{u}, \hat{v})$ is a strict subsolution of (1.5) for $\lambda_i + \mu_i$ large for $i = 1, 2$.

Next, we construct a strict supersolution $(\overline{u}_s, \overline{v}_s)$. We set $(\overline{u}_s, \overline{v}_s) = (\epsilon\phi_p, \epsilon\phi_q)$ for some $\epsilon > 0$ to be fixed below and ϕ_m is the eigenfunction corresponding the principal eigenvalue ν_m as discussed in earlier section with $m = p, q$. Then using the weak formulation of (5.2), we obtain

$$\begin{aligned}
\int_\Omega |\overline{u}_s|^{p-2}\nabla\overline{u}_s \cdot \nabla\xi \mathrm{d}x &= \int_\Omega |\epsilon\phi_p|^{p-2}\nabla(\epsilon\phi_p) \cdot \nabla\xi \mathrm{d}x = \nu_p \int_\Omega (\epsilon\phi_p)^{p-1}\xi \,\mathrm{d}x\,, \quad \text{and} \\
\int_\Omega |\overline{v}_s|^{q-2}\nabla\overline{v}_s \cdot \nabla\xi \mathrm{d}x &= \int_\Omega |\epsilon\phi_q|^{q-2}\nabla(\epsilon\phi_q) \cdot \nabla\xi \mathrm{d}x = \nu_q \int_\Omega (\epsilon\phi_q)^{q-1}\xi \,\mathrm{d}x
\end{aligned}$$

for all $\xi \in W$. Hence in order to establish that $(\epsilon\phi_p, \epsilon\phi_q)$ is a strict supersolution, we must show that

$$\begin{aligned}
\nu_p \int_\Omega (\epsilon\phi_p)^{p-1}\xi \,\mathrm{d}x &> \int_\Omega \left[\lambda_1 f_1(\epsilon\phi_p) + \mu_1 g_1(\epsilon\phi_q)\right]\xi \,\mathrm{d}x, \quad \text{and} \\
\nu_q \int_\Omega (\epsilon\phi_q)^{q-1}\xi \,\mathrm{d}x &> \int_\Omega \left[\lambda_2 f_2(\epsilon\phi_p) + \mu_2 g_2(\epsilon\phi_q)\right]\xi \,\mathrm{d}x
\end{aligned}$$

for all $\xi \in W$. That is, we must show

$$\begin{aligned}
\nu_p(\epsilon\phi_p)^{p-1} &> \lambda_1 f_1(\epsilon\phi_p) + \mu_1 g_1(\epsilon\phi_q)\,, \quad \text{and} \\
\nu_q(\epsilon\phi_p)^{q-1} &> \lambda_2 f_2(\epsilon\phi_q) + \mu_2 g_2(\epsilon\phi_q)
\end{aligned}$$

for some $\epsilon > 0$ fixed. Let K_1, $K_2 > 0$ be constants such that $\phi_p \le K_1\phi_q$ and $\phi_q \le K_2\phi_p$ in Ω. Then since g_1 and f_2 are nondecreasing, it suffices to show that

$$\nu_p(\epsilon\phi_p)^{p-1} > \lambda_1 f_1(\epsilon\phi_p) + \mu_1 g_1(K_2\epsilon\phi_p) \tag{5.11}$$

$$\nu_q(\epsilon\phi_q)^{q-1} > \lambda_2 f_2(\epsilon\phi_q) + \mu_2 g_2(K_1\epsilon\phi_q)\,. \tag{5.12}$$

First we establish (5.11). To this end, let

$$H_p(s) := \nu_p s^{p-1} - \lambda_1 f_1(s) - \mu_1 g_1(K_2 s)$$

for $s \geq 0$. By (C1) we get $H_p(0) = 0$. We also have $H_p'(s) = \nu_p(p-1)s^{p-2} - \lambda_1 f_1'(s) - K_2\mu_1 g_1'(s)$. Therefore, since $f_1'(0) < 0$ and $g_1'(0) = 0$ by our assumption (C1), we have

$$H_p'(0) = \begin{cases} -\lambda_1 f_1'(0) > 0 \ \text{ if } \ p > 2 \\ \nu_2 - \lambda_1 f_1'(0) > 0 \ \text{ if } \ p = 2 \\ \lim\limits_{s \to 0^+} H_p'(s) = +\infty \text{ if } 1 < p < 2. \end{cases}$$

Therefore, there exists $s_1 > 0$ such that $H_p(s) > 0$ for $s \in (0, s_1]$, that is, $\nu_p s^{p-1} > \lambda_1 f_1(s) + \mu_1 g_1(K_2 s)$ for $s \in (0, s_1]$. Similarly by setting

$$H_q(s) := \nu_q s^{q-1} - \lambda_2 f_2(K_1 s) - \mu_2 g_1(K_1 s)$$

we can show that there exists $s_2 > 0$ such that $H_q(s) > 0$ for $s \in (0, s_2]$. Hence $\nu_q s^{q-1} > \lambda_2 f_2(K_1 s) + \mu_2 g_2(K_1 s)$ for $s \in (0, s_2]$. Let $s^* := \min\{s_1, s_2\}$. Then there exists $\epsilon \in (0, s^*]$ so that (5.11) and (5.12) hold, that is, $(\overline{u}_s, \overline{v}_s) = (\epsilon\phi_p, \epsilon\phi_q)$ is a strict supersolution of (1.5) for $\epsilon \in (0, s^*]$.

Now we verify that the sub and supersolutions constructed above satisfy the hypotheses of Proposition 2.2. Clearly $(\underline{u}, \underline{v}) = (0,0) \leq (\epsilon\phi_p, \epsilon\phi_q) = (\overline{u}_s, \overline{v}_s)$ and $(\underline{u}, \underline{v}) = (0,0) \leq (\hat{u}, \hat{v}) = (\underline{u}_s, \underline{v}_s)$.

Next, $(\overline{u}_s, \overline{v}_s) = (\epsilon\phi_p, \epsilon\phi_q) \leq \left(Ce_p, [(\lambda_2 + \mu_2) f_2(C\|e_p\|_\infty)]^{\frac{1}{q-1}} e_q\right) = (\overline{u}, \overline{v})$ and

$(\underline{u}_s, \underline{v}_s) = (\hat{u}, \hat{v}) \leq \left(Ce_p, [(\lambda_2 + \mu_2) f_2(C\|e_p\|_\infty)]^{\frac{1}{q-1}} e_q\right) = (\overline{u}, \overline{v})$ by choosing C even larger. Finally, we choose $\epsilon > 0$ sufficiently small so that $(\underline{u}_s, \underline{v}_s) = (\hat{u}, \hat{v}) \not\leq (\epsilon\phi_p, \epsilon\phi_q) = (\overline{u}_s, \overline{v}_s)$. Then, by Proposition 2.2, there exist $(u_1, v_1) \in [(\underline{u}, \underline{v}), (\overline{u}_s, \overline{v}_s)]$, $(u_2, v_2) \in [(\underline{u}_s, \underline{v}_s), (\overline{u}, \overline{v})]$ and $(u_3, v_3) \in [(\underline{u}, \underline{v}), (\overline{u}, \overline{v})] \backslash \{(\underline{u}, \underline{v}), (\overline{u}_s, \overline{v}_s)] \cup [(\underline{u}_s, \underline{v}_s), (\overline{u}, \overline{v})\}$. However, since $(\underline{u}, \underline{v}) = (0, 0)$ is a solution, we cannot guarantee a positive solution in $[(\underline{u}, \underline{v}), (\overline{u}_s, \overline{v}_s)]$. In any case (1.5) has at least two positive solutions (u_2, v_2) and (u_3, v_3) for $\lambda_i + \mu_i$ large for $i = 1, 2$. This completes the proof of Theorem 1.5.

5.3 Proof of Theorem 1.6

We prove the nonexistence result by contradiction. Suppose $(u, v) \neq (0, 0)$ with $u \geq 0$ and $v \geq 0$ is a solution of (1.5) with $(u, v) \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega}) \times W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$. Using the weak formulation, we see that u satisfies

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \xi \mathrm{d}x = \lambda_1 \int_\Omega f_1(u) \xi \mathrm{d}x + \mu_1 \int_\Omega \widetilde{g_1}(v) \xi \, \mathrm{d}x$$

for all $\xi \in W$. In particular, it must hold true for $\xi = u$. Therefore

$$\int_\Omega |\nabla u|^p \mathrm{d}x = \lambda_1 \int_\Omega f_1(u) u \mathrm{d}x + \mu_1 \int_\Omega \widetilde{g_1}(v) u \, \mathrm{d}x \, .$$

Hypotheses on the nonlinearities f_1 and $\widetilde{g_1}$ yields

$$\int_\Omega |\nabla u|^p \mathrm{d}x \leq \lambda_1 A_1 \int_\Omega u^p \mathrm{d}x + \mu_1 B_1 \int_\Omega v^{p-1} u \, \mathrm{d}x \, . \tag{5.13}$$

But using the variational characterization of ν_p, the principal eigenvalue of (5.2)

$$\nu_p = \inf_{\psi \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla \psi|^p \, \mathrm{d}x}{\int_\Omega \psi^p \, \mathrm{d}x}$$

we obtain the following inequality

$$\nu_p \int_\Omega u^p \, \mathrm{d}x \leq \lambda_1 A_1 \int_\Omega u^p \mathrm{d}x + \mu_1 B_1 \int_\Omega v^{p-1} u \, \mathrm{d}x \, . \tag{5.14}$$

Similarly, we obtain

$$\nu_p \int_\Omega v^p \, \mathrm{d}x \leq \lambda_2 A_2 \int_\Omega v u^{p-1} \mathrm{d}x + \mu_2 B_2 \int_\Omega v^p \, \mathrm{d}x \, . \tag{5.15}$$

To simplify the terms involving vu^{p-1} and uv^{p-1}, we utilize the following inequality for $a > 0$ and $b > 0$

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}; \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Letting $a = v$ and $b = u^{p-1}$, it follows that $vu^{p-1} \leq \frac{v^p}{p} + \frac{u^p}{p'}$. Similarly $a = u$, $b = v^{p-1}$ yields $uv^{p-1} \leq \frac{u^p}{p} + \frac{v^p}{p'}$. This simplifies (5.14) to

$$\frac{\nu_p - \lambda_1 A_1}{\mu_1 B_1} \int_\Omega u^p \, \mathrm{d}x \leq \int_\Omega \left[\frac{u^p}{p} + \frac{v^p}{p'} \right] \, \mathrm{d}x \tag{5.16}$$

and (5.15) to

$$\frac{\nu_p - \mu_2 B_2}{\lambda_2 A_2} \int_\Omega v^p \, \mathrm{d}x \leq \int_\Omega \left[\frac{u^p}{p'} + \frac{v^p}{p} \right] \, \mathrm{d}x \, . \tag{5.17}$$

Adding (5.16) and (5.17) and using the fact that $\frac{1}{p} + \frac{1}{p'} = 1$, we get

$$\frac{\nu_p - \lambda_1 A_1}{\mu_1 B_1} \int_\Omega u^p \,\mathrm{d}x + \frac{\nu_p - \mu_2 B_2}{\lambda_2 A_2} \int_\Omega v^p \,\mathrm{d}x \leq \int_\Omega u^p \,\mathrm{d}x + \int_\Omega v^p \,\mathrm{d}x$$

that is,

$$\frac{\nu_p - \lambda_1 A_1 - \mu_1 B_1}{\mu_1 B_1} \int_\Omega u^p \,\mathrm{d}x + \frac{\nu_p - \lambda_2 A_2 - \mu_2 B_2}{\lambda_2 A_2} \int_\Omega v^p \,\mathrm{d}x \leq 0\,.$$

This gives a contradiction if $\max\{\lambda_1 A_1 + \mu_1 B_1, \lambda_2 A_2 + \mu_2 B_2\} < \nu_p$. Therefore there is no nonnegative solution to (1.5) if $\max\{\lambda_1 A_1 + \mu_1 B_1, \lambda_2 A_2 + \mu_2 B_2\} < \nu_p$. This completes the proof.

CHAPTER VI

COMPUTATIONAL RESULTS

In this Chapter we present some computational results for the boundary value problems that are related to the theoretical results obtained in this thesis. We use shooting method and Mathematica to investigate positive solutions of these problems. This method reduces the investigation of solutions of boundary value problems to the investigation of an initial value problem that satisfies the boundary conditions (shooting method).

We find solutions of single equation as well as of systems of two equations. We will describe the method for single equation at first. For example, the investigation of positive solutions of the boundary value problem

$$\begin{cases} -u''(r) = \lambda h(r) f(u(r)) \quad \text{for } 0 < r < 1; \\ u(0) = 0 = u(1), \end{cases} \tag{6.1}$$

where $h(r)$ is a weight function, is achieved by studying the positive solutions of the initial value problem

$$\begin{cases} -u''(r) = \lambda h(r) f(u(r)) \quad \text{for } 0 < r < 1; \\ u(0) = 0, \\ u'(0) = \alpha, \end{cases} \tag{6.2}$$

satisfying the boundary condition $u(1) = 0$.

We choose a range $R = [\alpha_{min}, \alpha_{max}]$ of initial conditions $u'(0) = \alpha$ within which we search for solutions satisfying the Dirichlet boundary condition $u(1) = 0$. The choice of this initial range is made heuristically, based on experience. This range is discretized and the numerical solution is found for each shooting angle within R. Then the solution will be verified for boundary condition. After determining the shooting angle we now solve the differential equation and find the supremum norm of the solution. Then we plot $\|u\|_\infty$ versus λ to obtain the bifurcation diagram. For solving the initial value problem we use NDSolve, a numerical solver from Mathematica.

We employ the following algorithms to carry out the computations. We use Algorithm 6.1 to investigate single equations.

Algorithm 6.1. *Given f and h*

Loop iteration... for $\lambda \in [0, \lambda_{max}]$;

    Loop iteration... $\alpha \in [0, \alpha_{max}]$;

        with $u(0) = 0, u'(0) = \alpha$, solve the initial value problem;

        for the solution u evaluate $u(1)$;

        if $u(1) = 0$, then store (contour) the point (λ, α);

For the stored (contour) points (λ, α), we compute $\|u\|_\infty$ for each solution u;

Now we explain our method for solving system of two equations. In order to describe our method for determining bifurcation diagram for systems, let us consider for example, the following boundary value problem

$$\begin{cases} -u''(r) = \lambda h_1(r) f(v(r)) & \text{for } 0 < r < 1; \\ -v''(r) = \lambda h_2(r) g(u(r)) & \text{for } 0 < r < 1; \\ u(0) = 0 = v(0); & \\ u(1) = 0 = v(1). & \end{cases} \tag{6.3}$$

Positive solutions of this boundary value problem are solutions of

$$\begin{cases} -u''(r) = \lambda h_1(r) f(v(r)) & \text{for } 0 < r < 1; \\ -u''(r) = \lambda h_2(r) g(u(r)) & \text{for } 0 < r < 1; \\ u(0) = v(0) = 0; & \\ u'(0) = \alpha, v'(0) = \beta. & \end{cases} \tag{6.4}$$

satisfying the boundary conditions $u(1) = 0 = v(1)$.

Let $U(\alpha, \beta) = u(1)$ and $V(\alpha, \beta) = v(1)$ corresponding to initial values $u'(0) = \alpha$ and $v'(0) = \beta$ respectively. Clearly the points (α_0, β_0) which satisfy $U(\alpha_0, \beta_0) = 0$ and $V(\alpha_0, \beta_0) = 0$ are the shooting angles for which we get Dirichlet boundary conditions $u(1) = 0$ and $v(1) = 0$. We choose a range $R_\alpha \times R_\beta$ of initial conditions within which we search for solutions satisfying the Dirichlet boundary conditions $u(1) = 0$ and $v(1) = 0$. Based on experience we choose the initial rectangular range heuristically. The rectangle is discretized and the numerical solution found for each pair of shooting angle will be verified for boundary conditions.

For example, let us consider $f(v(r)) = (v(r))^3 - 0.01$, $g(u(r)) = (u(r))^2 - 0.01$, $h_1(r) = r^{\frac{-1}{3}}$ and $h_2(r) = r^{\frac{-1}{2}}$. For visualizing, we plotted zero contour lines of functions $U(\alpha, \beta)$ and $V(\alpha, \beta)$. Their intersections are the points (α_0, β_0) for which the Dirichlet boundary conditions are satisfied. For $\lambda = 100$, we have the following zero contours shown in Figure 7. At the intersections are solutions (u, v) of the boundary value problem.

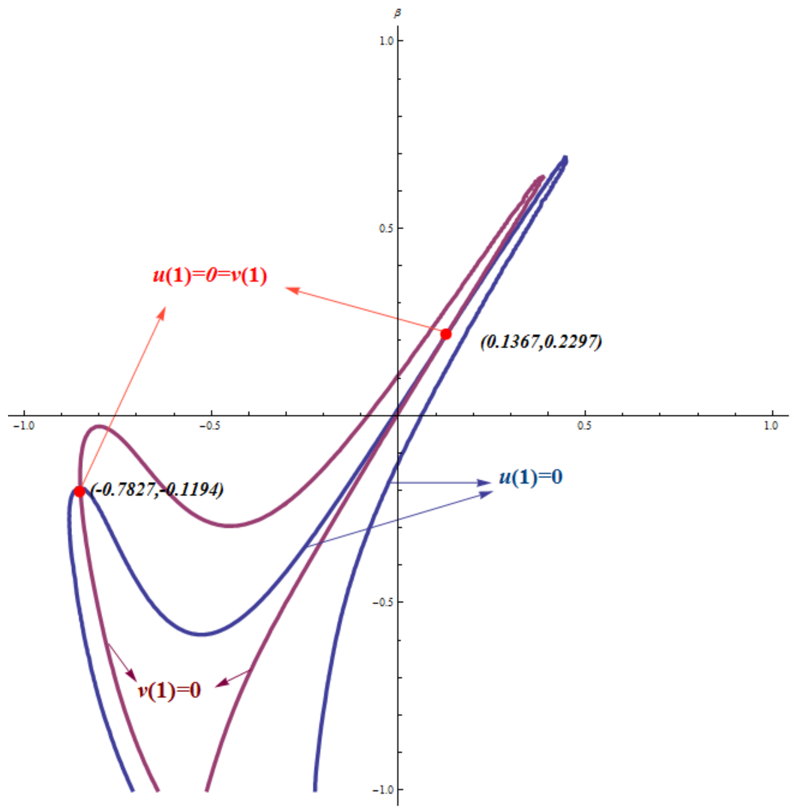

Figure 7. Solution contour at $\lambda = 100$ for system (6.4)

However, since we are studying nonnegative solutions, our solutions pair is only on the right hand side of Figure 8.

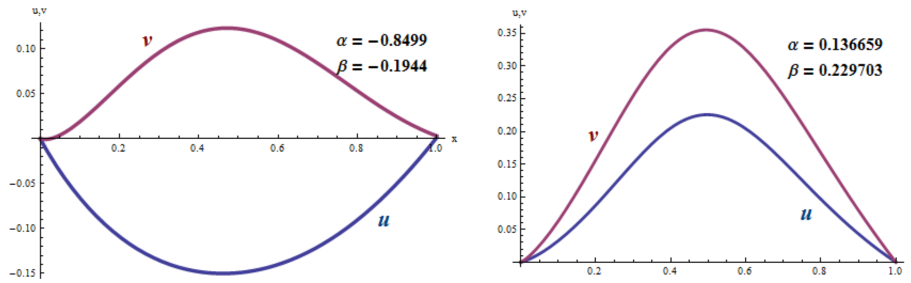

Figure 8. Solutions of system (6.4) for $\lambda = 100$

For solving systems of two equations we use the following Algorithm 6.2 , based on the root solving numerical technique in [Ham86].

Algorithm 6.2. *Loop iteration,* $\lambda \in [0, \lambda_{max}]$*;*

set up initial conditions u(0)=0=v(0);

set $u'(0) = \alpha$, $v'(0) = \beta$*;*

Loop iteration, $\alpha \in [0, \alpha_{max}]$, $\beta \in [0, \beta_{max}]$*;*

with initial values $u(0) = 0$, $u'(0) = \alpha$, $v(0) = 0$ *and* $v'(0) = \beta$*;*

setup and solve the initial value problem;

plot the contours satisfying the boundary values $u(1) = 0$ *and* $v(1) = 0$*;*

find the root of $u(1) = 0 = v(1)$ *(intersection of the above contours);*

6.1 Results Related to Theorem 1.1

We first investigate a result in single equation case corresponding to (1.1). In particular, we consider

$$\begin{cases} -(r^{N-1}|u'|^{p-2}u')' = \lambda r^{N-1}(u^\sigma - \epsilon) \quad \text{for } 0 < r < 1; \\ u'(0) = 0; \\ u(1) = 0. \end{cases} \tag{6.5}$$

We take $N = 3$, $\sigma = 1.4$, $\epsilon = 0.1$ and $p = 2.2$ and write a Mathematica program (Code 1 in Appendix) to implement the Algorithm 6.1. The resulting bifurcation diagram (shown in Figure 9) shows that (6.5) has a positive solution for $0 < \lambda \leq \lambda^*$ with $\lambda^* = 12.84$ and no positive solution for $\lambda > \lambda^*$. Moreover $\|u\|_\infty \to \infty$ as $\lambda \to 0$. This numerical result confirms the existence result in [AAP00], [JS04] and the nonexistence result in [CG06].

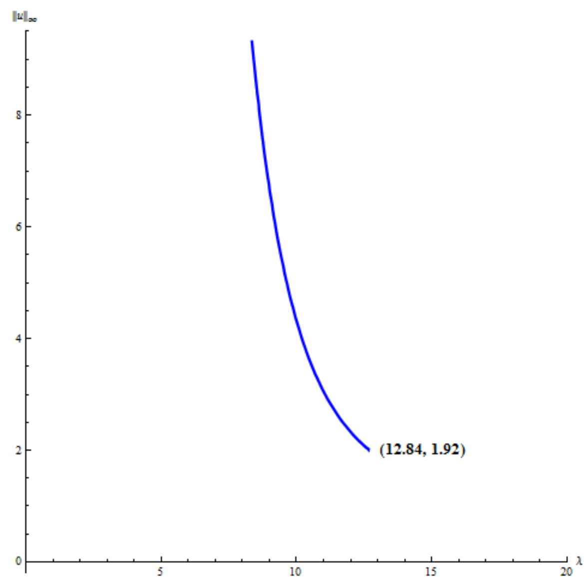

Figure 9. Bifurcation diagram for $\sigma = 1.4$, $p = 2.2$ and $N = 3$

Next we discuss the dependence of λ^* on the dimension N. This is illustrated in Figure 10. For the example above, if

(1) $N = 3$, then $\lambda^* = 12.84$,

(2) $N = 2$, then $\lambda^* = 12.01$, and

(3) $N = 1$, then $\lambda^* = 3.109$.

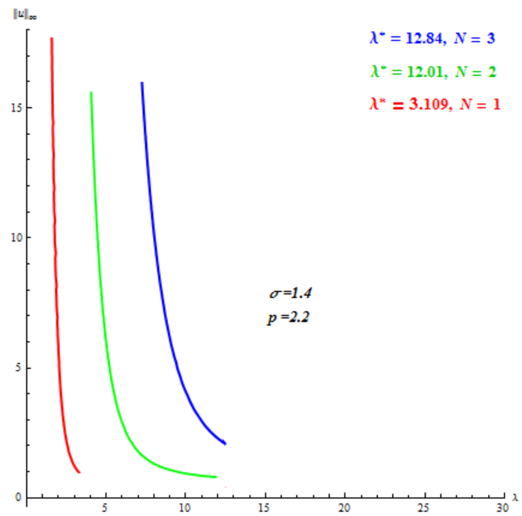

Figure 10. Bifurcation diagram for $\sigma = 1.4$ and $p = 2.2$

Next we discuss the dependence of λ^* on the exponent σ, as illustrated in Figure 11. It shows that for $N = 3$, $\epsilon = 0.1$ and $p = 2.2$ fixed, if

(1) $\sigma = 1.25$, then $\lambda^* = 20.15$,

(2) $\sigma = 1.3$, then $\lambda^* = 18.25$,

(3) $\sigma = 1.4$, then $\lambda^* = 12.84$, and

(4) $\sigma = 1.42$, then $\lambda^* = 11.3$.

This numerical result suggests that as σ increases, the range of λ for which the problem has a positive solution decreases.

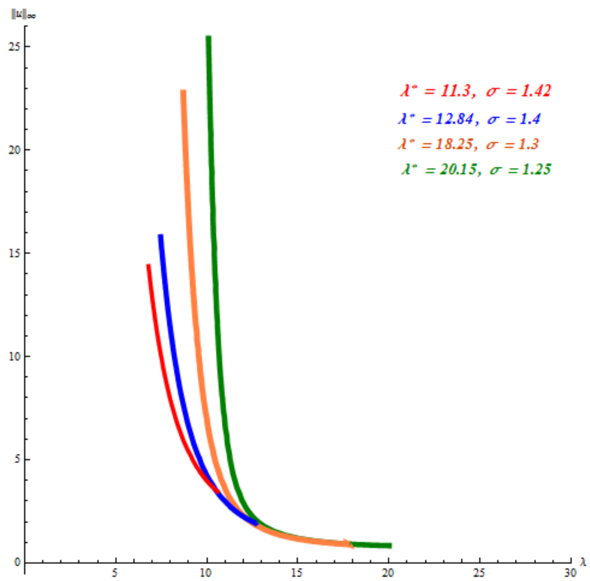

Figure 11. Bifurcation diagram for $p = 2.2$ and $N = 3$

The problem has positive solution for $0 < \lambda < 14.3 = \lambda^*$ and no positive solution for $\lambda \geq \lambda^* = 14.3$. Observe that for $\lambda = 14.2$, $u(1) \neq 0$ and $v(r) < 0$ for some $r \in (0, 1)$.

Now we discuss an example related to Theorem 1.1. We take $f(v) = v^{1.4} - 0.01$, $g(u) = u^{1.6} - 0.02$ with $p = q = 2.2$, and $N = 2$ in (1.1). We obtain the bifurcation diagram, Figure 12, which shows that (1.1) has a positive solution for $0 < \lambda < 14.4 = \lambda^*$ and no positive solution for $\lambda \geq \lambda^* = 14.4$. Observe that for $\lambda = 14.4$, $u(1) \neq 1$ and $v(1) \neq 1$. Figure 12 also depicts solution pairs (u, v) corresponding to $\lambda = 7$ and $\lambda = 14.3$. We implemented Algorithm 6.2 using Code 2 with $p \neq q$ (see Appendix) to generate Figure 12.

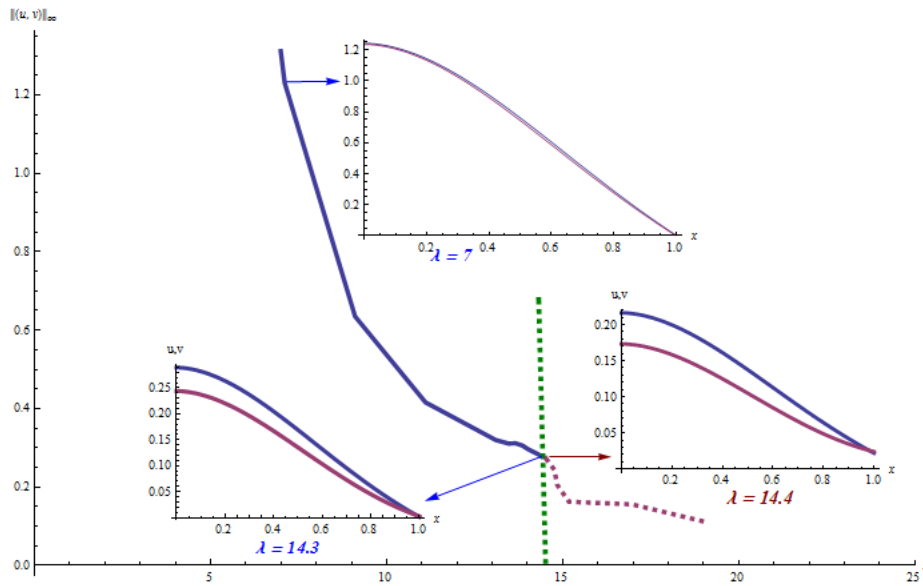

Figure 12. Bifurcation diagram depicting result of Theorem 1.1

6.2 Results Related to Theorem 1.2 and Theorem 1.3

First we consider the following single equation case. Take $f(u) = u^\sigma - \epsilon$ and $h_1(x) = x^{\frac{-1}{3}}$. Figure 13 shows the bifurcation diagram related to Theorem 1.3 for $\epsilon_1 = 0.01$. The diagrams are obtained by implementing Algorithm 6.1 and using Code 2 (see Appendix).

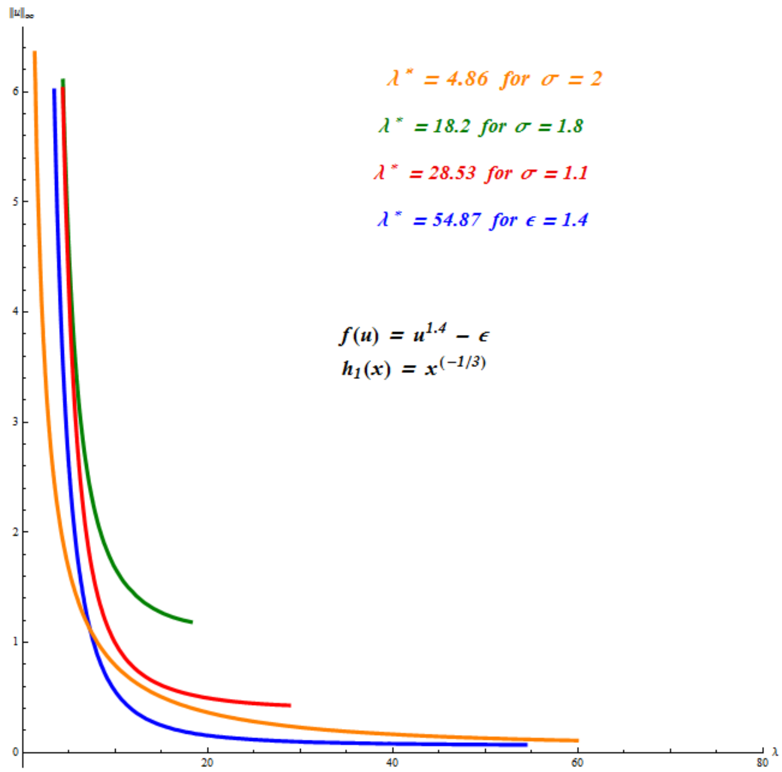

Figure 13. Bifurcation diagram for (1.4) with variation on σ

The diagram shows effect of varying σ on λ^*.

(1) For $\sigma = 1.1$, there exists a positive solution for $0 < \lambda < \lambda^* = 17.6$ and no positive solution for $\lambda > \lambda^* = 17.6$.

(2) For $\sigma = 1.4$, there exists a positive solution for $0 < \lambda < \lambda^* = 54.66$ and no positive solution for $\lambda > \lambda^* = 54.66$.

(3) For $\sigma = 1.8$, there exists a positive solution for $0 < \lambda < \lambda^* = 91.91$ and no positive solution for $\lambda > \lambda^* = 17.6$.

(4) For $\sigma = 2$, there exists a positive solution for $0 < \lambda < \lambda^* = 109.6$ and no positive solution for $\lambda > \lambda^* = 109.6$.

Next we illustrate the effect of varying ϵ on λ^* in Figure 14.

(1) For $\epsilon = 0.01$, there exists a positive solution for $0 < \lambda < \lambda^* = 74.25$ and no positive solution for $\lambda > \lambda^* = 74.25$.

(2) For $\epsilon = 0.1$, there exists a positive solution for $0 < \lambda < \lambda^* = 31.12$ and no positive solution for $\lambda > \lambda^* = 31.12$.

(3) For $\epsilon = 0.5$, there exists a positive solution for $0 < \lambda < \lambda^* = 18.37$ and no positive solution for $\lambda > \lambda^* = 18.37$.

(4) For $\epsilon = 1$, there exists a positive solution for $0 < \lambda < \lambda^* = 13.19$ and no positive solution for $\lambda > \lambda^* = 13.19$.

These results were obtained (for $\sigma = 1.4$) by using Algorithm 6.1 and Code 2 of Appendix.

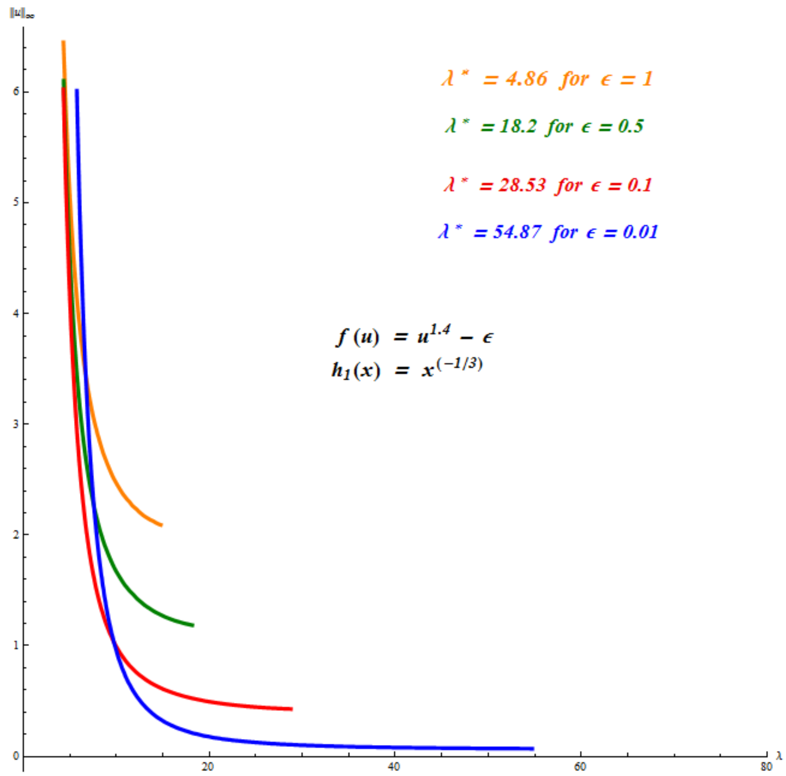

Figure 14. Bifurcation diagram for (1.4) with variation on ϵ

Our next computational result deals with the system (1.3). Here we take $f(v) = v^3 - 0.01$, $g(u) = u^2 - 0.01$, $h_1(x) = x^{\frac{-1}{3}}$ and $h_2(x) = x^{\frac{-1}{2}}$. By implementing Algorithm 6.2 and Code 3 (see Appendix) we obtain the following bifurcation diagram, Figure 15, confirming results of Theorem 1.2. In particular it shows that (1.3) has a positive solution for $0 < \lambda < 106.6$ and no positive solution for $\lambda > 117$. The diagram also depicts positive solutions corresponding to $\lambda = 20.$ and $\lambda = 117$.

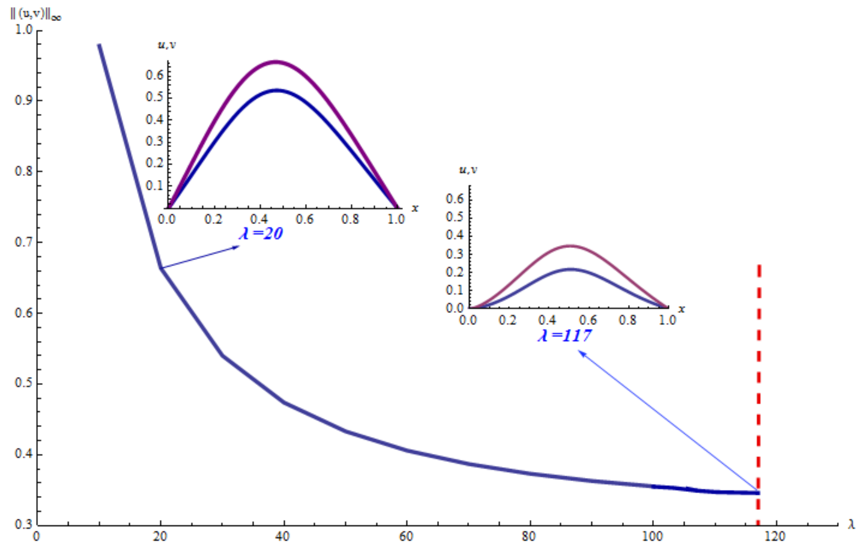

Figure 15. Bifurcation diagram depicting results of Theorem 1.2

6.3 Results Related to Theorem 1.4 and Theorem 1.6

First, we discuss our computational result for the infinite semipositone single equation case problem

$$\begin{cases} -(r^{N-1}|u'|^{p-2}u')' = \lambda r^{N-1}\frac{u^{1.2}-0.5}{u^{1/3}} \quad \text{for } 0 < r < 1; \\ u'(0) = 0; \\ u(1) = 0, \end{cases} \tag{6.6}$$

corresponding to Theorem 1.4. We take $p = 2.4$ and $N = 2$. We implemented Algorithm 6.1 and appropriately modified Code 1 to obtain Figure 16.

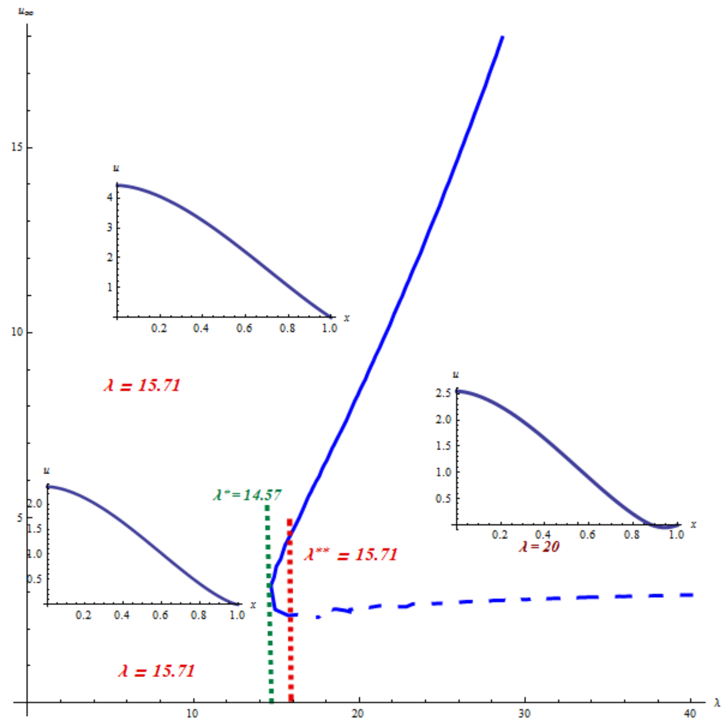

Figure 16. Bifurcation diagram infinite semipositone problem

The bifurcation diagram shows that

(1) for $\lambda < \lambda^* = 14.57$ there is no positive solution,

(2) for $\lambda = 14.57$ there is one positive solutions,

(3) for $\lambda^* < \lambda < \lambda^{**} = 15.71$ there are two positive solutions, and

(4) for $\lambda > \lambda^{**} = 15.71$ there is one positive solution.

Our last computational result deals with system (1.5). To simplify our computation we assume $\lambda_1 = 0 = \mu_2$ and $\mu_1 = \lambda_2 = \lambda$. We take $N = 2$, $p = 2$, $q = 2.2$, $\widetilde{f}(v) = \frac{v^{0.8}-0.01}{v^{\frac{2}{3}}}$ and $\widetilde{g}(u) = \frac{u-0.02}{u^{\frac{1}{2}}}$. The bifurcation diagram is shown in Figure 17. It shows that (1.5) has no positive solution for $0 < \lambda < 0.6$ and a positive solution for $\lambda \geq 0.6$. Here we implemented Algorithm 6.2, and Code 4.

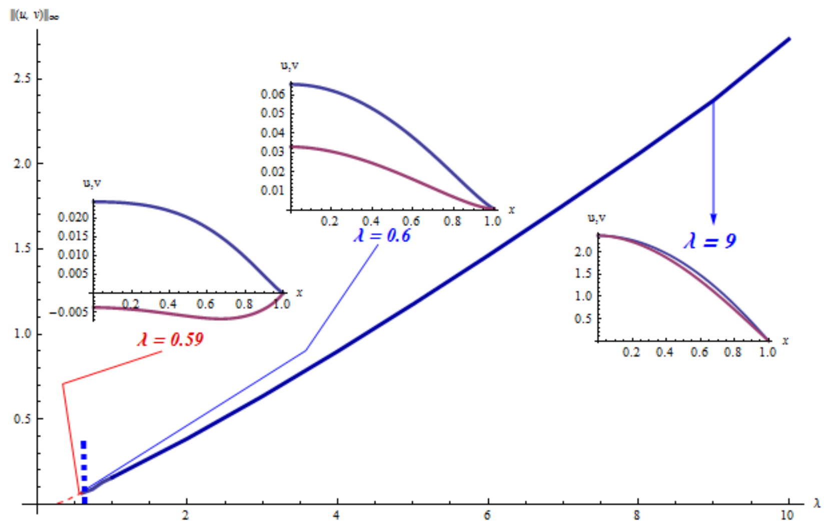

Figure 17. Bifurcation diagram depicting results of Theorem 1.4

Here we make the following remark about our method to determine the value of λ^* (in Figure 17) at which there exist a solution for $\lambda \geq \lambda^*$. We find the solution satisfying the Dirichlet boundary condition and draw the contour of such solutions for each function u, v. The contour is sketched over the rectangular discretized range of $[0, \alpha_{max}] \times [0, \beta_{max}]$. Here the solutions are found by considering the initial conditions $u'(0) = \alpha$ and $v'(0) = \beta$, with $\alpha \in [0, \alpha_{max}]$ and $\beta \in [0, \beta_{max}]$.

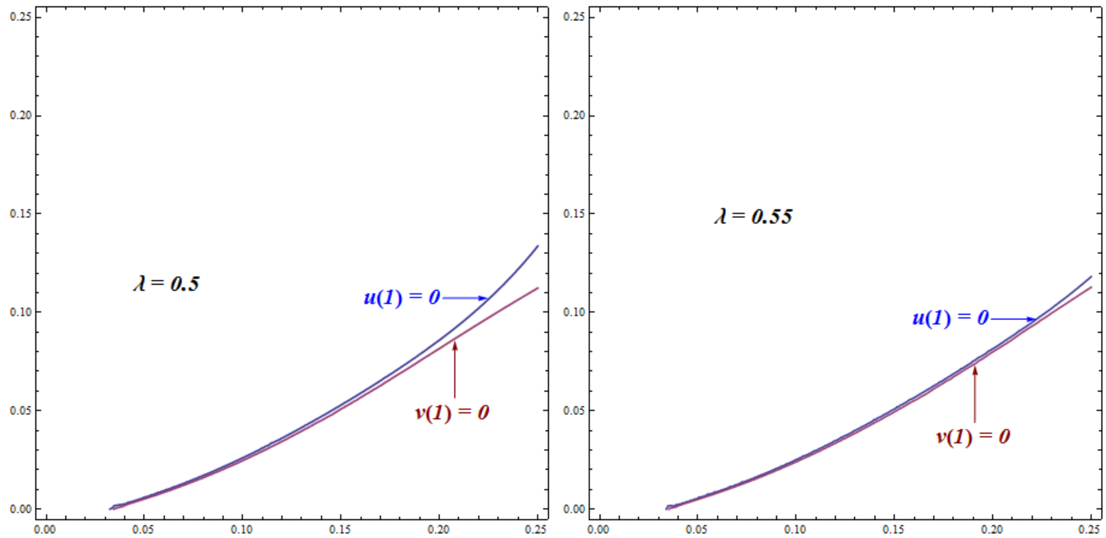

Figure 18. Iteration showing no solution for $\lambda = 0.5$ and $\lambda = 0.55$

We observe that the contours does not intersect to give a pair of solution satisfying the boundary condition (see Figure 18 and Figure 19).

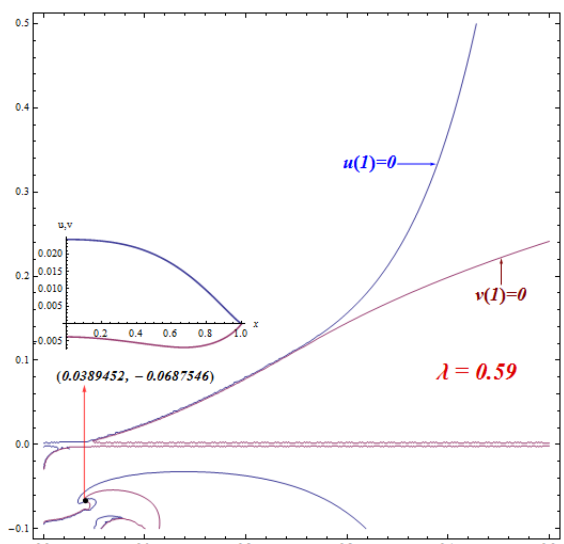

Figure 19. Iteration showing no positive solution for $\lambda = 0.599$

But as we see in Figure 20 for $\lambda \geq 0.6$ the two contours intersect.

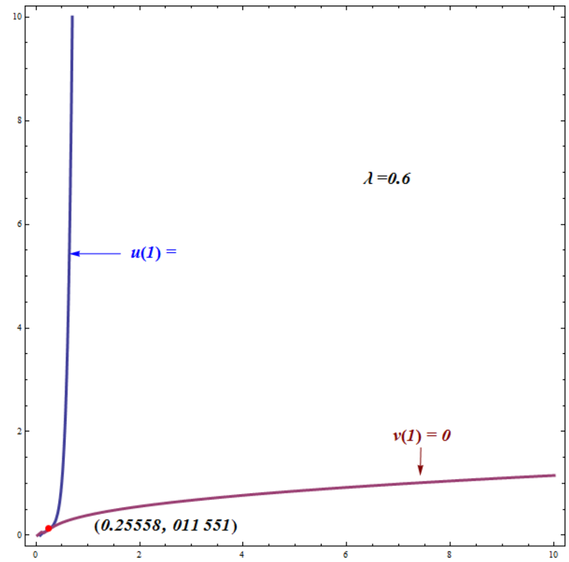

Figure 20. Iteration showing positive solutions for $\lambda = 0.6$

For instance, in Figure 21 we see that for $\lambda = 7$ the two contours intersect.

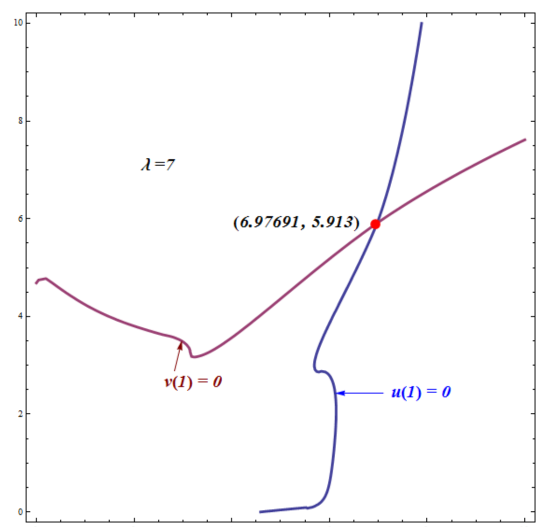

Figure 21. Iteration showing positive solutions for $\lambda = 7$

CHAPTER VII

CONCLUSIONS AND FUTURE DIRECTIONS

7.1 Conclusion

In this thesis we studied positive solutions of elliptic systems with nonlinearities satisfying semipositone behavior at the origin. In particular we discussed existence, nonexistence and multiplicity results with respect to a parameter or multiparameters. We employed methods such as degree theory, sub - super solutions method and energy analysis to establish results. Finally we use computational technique to compute bifurcation diagrams of solution with respect to the parameters.

7.2 Future Directions

We wish to investigate the following open problems in the future.

(1) Establish that a nonnegative solution of (1.1) is positive, radially symmetric and radially decreasing.

(2) Investigate Theorem 1.1 in nonradial bounded domains.

(3) Establish Theorem 1.1 for $pq-$ Laplacian systems.

(4) Extend Theorem 1.3 to systems.

(5) Study exterior domain problems for non-radial solutions.

(6) Establish Theorem 1.6 for $p \neq q$.

REFERENCES

[AAB94] A. Ambrosetti, D. Arcoya, and B. Buffoni, *Positive solutions for some semi-positone problems via bifurcation theory*, Differential Integral Equations **7** (1994), no. 3-4, 655–663.

[AAP00] A. Ambrosetti, J. Garcia Azorero, and I. Peral, *Elliptic variational problems in $\mathbf{R}^N$ with critical growth*, J. Differential Equations **168** (2000), no. 1, 10–32, Special issue in celebration of Jack K. Hale's 70th birthday, Part 1 (Atlanta, GA/Lisbon, 1998).

[AC13] Abraham Abebe and Maya Chhetri, *Nonexistence of positive radial solutions for a class of quasilinear systems*, Submitted, 2013.

[ACS14] Abraham Abebe, Maya Chhetri, and R. Shivaji, *Positive solutions for a class of multiparameter elliptic systems*, Submitted, 2014.

[ACSS14] Abraham Abebe, Maya Chhetri, Lakshmi Sankar, and R. Shivaji, *Positive solutions for a class of superlinear semipositone systems on exterior domains*, Submitted, 2014.

[Ama76] Herbert Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev. **18** (1976), no. 4, 620–709.

[ANZ92] W. Allegretto, P. Nistri, and P. Zecca, *Positive solutions of elliptic non-positone problems*, Differential Integral Equations **5** (1992), no. 1, 95–101.

[Ari69] Rutherford Aris, *On stability criteria of chemical reaction engineering*, Chemical Engineering Science **24** (1969), no. 1, 149 – 169.

[AS07] Jaffar Ali and R. Shivaji, *Positive solutions for a class of p-Laplacian systems with multiple parameters*, J. Math. Anal. Appl. **335** (2007), no. 2, 1013–1019.

[BCN96] H. Berestycki, L. A. Caffarelli, and L. Nirenberg, *Inequalities for second-order elliptic equations with applications to unbounded domains. I*, Duke Math. J. **81** (1996), no. 2, 467–494, A celebration of John F. Nash, Jr.

[BCS89] K. J. Brown, Alfonso Castro, and R. Shivaji, *Nonexistence of radially symmetric nonnegative solutions for a class of semi-positone problems*, Differential Integral Equations **2** (1989), no. 4, 541–545.

[BIS81] K. J. Brown, M. M. A. Ibrahim, and R. Shivaji, *S-shaped bifurcation curves*, Nonlinear Anal. **5** (1981), no. 5, 475–486.

[Bro98] Friedemann Brock, *Radial symmetry for nonnegative solutions of semilinear elliptic equations involving the p-Laplacian*, Progress in partial differential equations, Vol. 1 (Pont-à-Mousson, 1997), Pitman Res. Notes Math. Ser., vol. 383, Longman, Harlow, 1998, pp. 46–57.

[BS79a] F. Brauer and A. C. Soudack, *Stability regions and transition phenomena for harvested predator-prey systems*, J. Math. Biol. **7** (1979), no. 4, 319–337.

[BS79b] ______, *Stability regions in predator-prey systems with constant-rate prey harvesting*, J. Math. Biol. **8** (1979), no. 1, 55–71.

[BS82] ______, *On constant effort harvesting and stocking in a class of predator-prey systems*, J. Theoret. Biol. **95** (1982), no. 2, 247–252.

[BS83] K. J. Brown and R. Shivaji, *Simple proofs of some results in perturbed bifurcation theory*, Proc. Roy. Soc. Edinburgh Sect. A **93** (1982/83), no. 1-2, 71–82.

[CD74] Jagdish Chandra and Paul Wm. Davis, *A monotone method for quasilinear boundary value problems*, Arch. Rational Mech. Anal. **54** (1974), 257–266.

[CD80] J. Chandra and P. W. Davis, *On the relation between monotone methods and temporal evolution in fully nonlinear boundary value problems*, J. Inst. Math. Appl. **25** (1980), no. 3, 231–240.

[CG06] Maya Chhetri and Petr Girg, *Nonexistence of nonnegative solutions for a class of $(p-1)$-superhomogeneous semipositone problems*, J. Math. Anal. Appl. **322** (2006), no. 2, 957–963.

[CG09] ______, *Existence and nonexistence of positive solutions for a class of superlinear semipositone systems*, Nonlinear Anal. **71** (2009), no. 10, 4984–4996.

[CG13] ______, *Existence of positive solutions for a class of superlinear semipositone systems*, J. Math. Anal. Appl. **408** (2013), no. 2, 781–788.

[CHS95] Alfonso Castro, M. Hassanpour, and R. Shivaji, *Uniqueness of nonnegative solutions for a semipositone problem with concave nonlinearity*, Comm. Partial Differential Equations **20** (1995), no. 11-12, 1927–1936.

[CL70] Donald S. Cohen and Theodore W. Laetsch, *Nonlinear boundary value problems suggested by chemical reactor theory.*, J. Differential Equations **7** (1970), 217–226.

[CMS01] Alfonso Castro, C. Maya, and R. Shivaji, *Positivity of nonnegative solutions for cooperative semipositone systems*, Dynamic systems and applications, Vol. 3 (Atlanta, GA, 1999), Dynamic, Atlanta, GA, 2001, pp. 113–119.

[CR73] Michael G. Crandall and Paul H. Rabinowitz, *Bifurcation, perturbation of simple eigenvalues and linearized stability*, Arch. Rational Mech. Anal. **52** (1973), 161–180.

[CS88] Alfonso Castro and R. Shivaji, *Nonnegative solutions for a class of nonpositone problems*, Proc. Roy. Soc. Edinburgh Sect. A **108** (1988), no. 3-4, 291–302.

[CSS12] Alfonso Castro, Lakshmi Sankar, and R. Shivaji, *Uniqueness of nonnegative solutions for semipositone problems on exterior domains*, J. Math. Anal. Appl. **394** (2012), no. 1, 432–437.

[CT00] Mabel Cuesta and Peter Takáč, *A strong comparison principle for positive solutions of degenerate elliptic equations*, Differential Integral Equations **13** (2000), no. 4-6, 721–746.

[dF94] Djairo G. de Figueiredo, *Monotonicity and symmetry of solutions of elliptic systems in general domains*, NoDEA Nonlinear Differential Equations Appl. **1** (1994), no. 2, 119–123.

[dFLN82] D. G. de Figueiredo, P.-L. Lions, and R. D. Nussbaum, *A priori estimates and existence of positive solutions of semilinear elliptic equations*, J. Math. Pures Appl. (9) **61** (1982), no. 1, 41–63.

[DMP03] Zdzisław Denkowski, Stanisław Migórski, and Nikolas S. Papageorgiou, *An introduction to nonlinear analysis: theory*, Kluwer Academic Publishers, Boston, MA, 2003.

[DOS06] H. Dang, S. Oruganti, and R. Shivaji, *Nonexistence of positive solutions for a class of semilinear elliptic systems*, Rocky Mountain J. Math. **36** (2006), no. 6, 1845–1855.

[DT98] Guoren Dai and Moxun Tang, *Coexistence region and global dynamics of a harvested predator-prey system*, SIAM J. Appl. Math. **58** (1998), no. 1, 193–210 (electronic).

[FG95] Irene Fonseca and Wilfrid Gangbo, *Degree theory in analysis and applications*, Oxford Lecture Series in Mathematics and its Applications, vol. 2, The Clarendon Press, Oxford University Press, New York, 1995, Oxford Science Publications.

[GNN79] B. Gidas, Wei Ming Ni, and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), no. 3, 209–243.

[Ham86] R. W. Hamming, *Numerical methods for scientists and engineers*, second ed., Dover Publications, Inc., New York, 1986.

[HS03] D. D. Hai and R. Shivaji, *Existence and uniqueness for a class of quasilinear elliptic boundary value problems*, J. Differential Equations **193** (2003), no. 2, 500–510.

[HSS12] D. D. Hai, Lakshmi Sankar, and R. Shivaji, *Infinite semipositone problems with asymptotically linear growth forcing terms*, Differential Integral Equations **25** (2012), no. 11-12, 1175–1188.

[JS04] Jon Jacobsen and Klaus Schmitt, *Radial solutions of quasilinear elliptic differential equations*, Handbook of differential equations, Elsevier/North-Holland, Amsterdam, 2004, pp. 359–435.

[KC67] Herbert B. Keller and Donald S. Cohen, *Some positone problems suggested by nonlinear heat generation*, J. Math. Mech. **16** (1967), 1361–1376.

[Kel69] Herbert B. Keller, *Elliptic boundary value problems suggested by nonlinear diffusion processes*, Arch. Rational Mech. Anal. **35** (1969), 363–381.

[KW75] Jerry L. Kazdan and F. W. Warner, *Remarks on some quasilinear elliptic equations*, Comm. Pure Appl. Math. **28** (1975), no. 5, 567–597.

[Lae71] Theodore Laetsch, *The number of solutions of a nonlinear two point boundary value problem*, Indiana Univ. Math. J. **20** (1970/1971), 1–13.

[Lio82] P.-L. Lions, *On the existence of positive solutions of semilinear elliptic equations*, SIAM Rev. **24** (1982), no. 4, 441–467.

[Llo78] N. G. Lloyd, *Degree theory*, Cambridge University Press, Cambridge-New York-Melbourne, 1978, Cambridge Tracts in Mathematics, No. 73.

[LSS11] Eun Kyoung Lee, Lakshmi Sankar, and R. Shivaji, *Positive solutions for infinite semipositone problems on exterior domains*, Differential Integral Equations **24** (2011), no. 9-10, 861–875.

[LSY09a] Eun Kyoung Lee, R. Shivaji, and Jinglong Ye, *Classes of infinite semipositone systems*, Proc. Roy. Soc. Edinburgh Sect. A **139** (2009), no. 4, 853–865.

[LSY09b] Eun Kyoung Lee, Ratnasingham Shivaji, and Jinglong Ye, *Subsolutions: a journey from positone to infinite semipositone problems*, Proceedings of the Seventh Mississippi State–UAB Conference on Differential Equations and Computational Simulations, Electron. J. Differ. Equ. Conf., vol. 17, Southwest Texas State Univ., San Marcos, TX, 2009, pp. 123–131.

[LSY10] Eun Kyoung Lee, R. Shivaji, and Jinglong Ye, *Classes of singular pq-Laplacian semipositone systems*, Discrete Contin. Dyn. Syst. **27** (2010), no. 3, 1123–1132.

[Maw99] Jean Mawhin, *Leray-Schauder degree: a half century of extensions and applications*, Topol. Methods Nonlinear Anal. **14** (1999), no. 2, 195–228.

[MGHN92] M. R. Myerscough, B. F. Gray, W. L. Hogarth, and J. Norbury, *An analysis of an ordinary differential equation model for a two-species predator-prey system with harvesting and stocking*, J. Math. Biol. **30** (1992), no. 4, 389–411.

[MS99] C. Maya and R. Shivaji, *Multiple positive solutions for a class of semilinear elliptic boundary value problems*, Nonlinear Anal. **38** (1999), no. 4, Ser. A: Theory Methods, 497–504.

[OSS02] Shobha Oruganti, Junping Shi, and Ratnasingham Shivaji, *Diffusive logistic equation with constant yield harvesting. I. Steady states*, Trans. Amer. Math. Soc. **354** (2002), no. 9, 3601–3619 (electronic).

[Pao92] C. V. Pao, *Nonlinear parabolic and elliptic equations*, Plenum Press, New York, 1992.

[Par74] Seymour V. Parter, *Solutions of a differential equation arising in chemical reactor processes*, SIAM J. Appl. Math. **26** (1974), 687–716.

[Per03] Kanishka Perera, *Multiple positive solutions for a class of quasilinear elliptic boundary-value problems*, Electron. J. Differential Equations (2003), No. 7, 5 pp. (electronic).

[Rab86] Paul H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conference Series in

Mathematics, vol. 65, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1986.

[Rab74] ______, *Pairs of positive solutions of nonlinear elliptic partial differential equations*, Indiana Univ. Math. J. **23** (1973/74), 173–186.

[Sat73] David H. Sattinger, *Topics in stability and bifurcation theory*, Lecture Notes in Mathematics, Vol. 309, Springer-Verlag, Berlin-New York, 1973.

[Sat72] D. H. Sattinger, *Monotone methods in nonlinear elliptic and parabolic boundary value problems*, Indiana Univ. Math. J. **21** (1971/72), 979–1000.

[Sel98] James F. Selgrade, *Using stocking or harvesting to reverse period-doubling bifurcations in discrete population models*, J. Differ. Equations Appl. **4** (1998), no. 2, 163–183.

[Shi87] Ratnasingham Shivaji, *A remark on the existence of three solutions via sub-super solutions*, Nonlinear analysis and applications (Arlington, Tex., 1986), Lecture Notes in Pure and Appl. Math., vol. 109, Dekker, New York, 1987, pp. 561–566.

[SSS13] Lakshmi Sankar, Sarath Sasi, and R. Shivaji, *Semipositone problems with falling zeros on exterior domains*, J. Math. Anal. Appl. **401** (2013), no. 1, 146–153.

[Str77] Walter A. Strauss, *Existence of solitary waves in higher dimensions*, Comm. Math. Phys. **55** (1977), no. 2, 149–162.

[SY11] R. Shivaji and Jinglong Ye, *Nonexistence results for classes of* 3×3 *elliptic systems*, Nonlinear Anal. **74** (2011), no. 4, 1485–1494. MR 2746826 (2011k:35046)

[Tam79] K. K. Tam, *Construction of upper and lower solutions for a problem in combustion theory*, J. Math. Anal. Appl. **69** (1979), no. 1, 131–145.

[Tro81] William C. Troy, *Symmetry properties in systems of semilinear elliptic equations*, J. Differential Equations **42** (1981), no. 3, 400–413.

[Uns88] Sumalee Unsurangsie, *Existence of a solution for a wave equation and an elliptic Dirichlet problem*, ProQuest LLC, Ann Arbor, MI, 1988, Thesis (Ph.D.)–University of North Texas.

APPENDIX A

CODE 1

p-Laplacian Equations in a Ball

```
ClearAll["Global`*"]; (*Clearing variables*)
s = 1.4;          (*nonlinearity degree *)
p = 2.2;          (*setting p*)
N = 3;            (*declaring dimension *)
H[L_?NumberQ, B_?NumberQ] := (*defining function H*)
  (
   u[1] /.        (*defining u[1]*)
     NDSolve
     [              (*calling solver NDSolve*)
       {
        (p - 1)*(Abs[u'[x]])^(p - 2)*u''[x] +
        L*(((Abs[u[x]])^s) - 0.1) +
        ((N - 1)/(Abs[x]))*u'[x]*(Abs[u'[x]])^(p - 2)
        == 0, (*setting the governing equation*)
        u[-1] == 0, u'[-1] == B (*initial conditions*)
       },
        u, {x, -1, 1}, PrecisionGoal -> Infinity
     ][[1]]
  );
graphics =
  ContourPlot
```

```
        [
         H[l, m], {l, 0, 20}, {m, 0, 100},
         Contours -> {0},
         ContourShading -> False
        ]; (*finding contour points*)
Hx[L_?NumberQ, B_?NumberQ] :=(*defining function Hx*)
  Block
    [ {x},              (*declaring local variable x*)
      (
        u /.            (*setting up u[x]*)
          NDSolve
            [       (*calling solver NDSolve*)
              {
               (p - 1)*(Abs[u'[x]])^(p - 2)*u''[x] +
               L*(((Abs[u[x]])^s) - 0.1) +
               ((N - 1)/(Abs[x]))*u'[x] *(Abs[u'[x]])^(p - 2)
               == 0,  (*setting the governing equation*)
               u[-1] == 0, u'[-1] == B (*initial conditions*)
              },
               u, {x, -1, 1}, PrecisionGoal -> Infinity
            ]
            [[1]]
      )
    ];
```

```
allLines = Cases
            [
              Normal[graphics],_Line, Infinity
            ]; (*getting lines from contours*)
myMax[t_, c_] :=
    Block
      [{temp = Hx[t, c]},
        Max[Table[Abs[temp[x]], {x, 0, 1, 0.01}]]
      ];
al = allLines /. {t_?NumberQ, c_?NumberQ} :>
  {c, myMax[t,c]}; (*setting SupNorm(u) vs L*)
Graphics
    [
     {Blue, al}, Axes -> True, AxesOrigin -> {0, 0},
      AxesLabel -> \{L,Subscript \LeftDoubleBracketingBar u
      \RightDoubleBracketingBar, Infinity\}, AspectRatio -> 1,
      PlotRange -> {{0, 30}, Automatic}
    ] (*plotting*)
```

APPENDIX B

CODE 2

Laplacian Equation in an Exterior Domain

```
ClearAll["Global`*"]; (*Clearing variables*)
s = 3;          (*nonlinearity degree *)
G[L_?NumberQ, A_?NumberQ] := (*defining function G*)
    (
     u[1] /.          (*defining u[1]*)
       NDSolve
        [       (*calling solver NDSolve*)
          {
           u''[x] +
           L*(0.0001+x^(-1/3))*((Abs[v[x]])^s - 0.01) == 0
           u[0]=0,u'[0]=A ,(*initial conditions*)
          },    (*setting governing equation*)
           u,\{x,0,1\}, PrecisionGoal -> Infinity
        ]
        [[1]]
    );
graphics =
   ContourPlot
      [
        G[l, m],{l, 0, 20},{m, 0, 250},
        Contours -> {0}, ContourShading -> False
```

```
        ];      (*finding contour points*)
Gx[L_?NumberQ, A_?NumberQ] :=    (*defining function Gx*)
   Block
     [ {x},               (*declaring local variable x*)
     (
      u /.            (*setting up u[x]*)
        NDSolve
         [        (*callig solver NDSolve*)
           {
            u''[x] +
            L*(0.0001+x^(-1/3))*((Abs[v[x]])^s - 0.01)== 0
            u[0]=0,u'[0]=A ,(*initial conditions*)
           },     (*setting governing equation*)
            u,\{x,0,1\},PrecisionGoal-> Infinity
         ]
         [[1]]
     );
allLines =
     Cases
         [
          Normal[graphics],_Line, Infinity
         ]; (*getting lines from contours*)
myMax[t_, c_] :=
   Block
```

```
      [
       {temp = Gx[t, c]},
       Max[Table[Abs[temp[x]], {x, 0, 1, 0.01}]]
      ];
allLines /. {t_?NumberQ, c_?NumberQ} :>
  {c, myMax[t, c]}; (*setting SupNorm (u) vs L*)
Graphics
  [
   {Blue, %}, Axes -> True, AxesOrigin -> {0, 0},
   AspectRatio -> 1, AxesLabel ->
   {L,Subscript \LeftDoubleBracketingBar
   u \RightDoubleBracketingBar, Infinity}
  ] (*plotting*)
```

APPENDIX C

CODE 3

pq- Laplacian System in a Ball

```
ClearAll["Global`*"];          (*Clearing variables*)
s = 1.4;            (*nonlinearity degree*)
N = 2;              (*setting dimension*)
d = 1.6;            (*nonlinearity degree*)
p = 2.2;            (*setting p*)
q = 2.2;            (*setting q*)
i = 1;              (*initializing i*)
m = 7;              (*setting max iteration*)
Array[init,m];  (*initializing array*)
Array[MM,m];  (*initializing array*)
For
 [
  L = 9, L < 20, L += 2,    (*setting a for loop*)
  Z[A_?NumericQ, B_?NumericQ] := (*defining function Z*)
    (
     u[1] /.      (*defining u[1]*)
       NDSolve
         [       (*calling solver NDSolve*)
          {
           (p - 1)*(Abs[u'[x]])^(p - 2)*u''[x] +
           ((N - 1)/Abs[x])*u'[x]*(Abs[u'[x]])^(p - 2)+
```

```
        L*((Abs[v[x]])^s - 0.01) == 0,
        (q - 1)*(Abs[v'[x]])^(q - 2)*v''[x] +
        ((N - 1)/Abs[x])*v'[x] *(Abs[v'[x]])^(q - 2) +
        L*((Abs[u[x]])^d - 0.02) == 0,
        u[-1] == 0 , u'[-1] == A,
        v[-1] == 0, v'[-1] == B     (*initial conditions*)
       },           (*setting governing equations*)
        {u, v}, {x, -1, 1}, PrecisionGoal-> Infinity
      ]
    [[1]]
  );
R[A_?NumericQ, B_?NumericQ] := (*defining function R*)
  (
   v[1] /.     (*defining v[1]*)
     NDSolve
      [      (*calling solver NDSolve*)
       {
        (p - 1)*(Abs[u'[x]])^(p - 2)*u''[x] +
        ((N - 1)/Abs[x])*u'[x]*(Abs[u'[x]])^(p - 2)+
        L*((Abs[v[x]])^s - 0.01) == 0,
        (q - 1)*(Abs[v'[x]])^(q - 2)*v''[x] +
        ((N - 1)/Abs[x])*v'[x] *(Abs[v'[x]])^(q - 2) +
        L*((Abs[u[x]])^d - 0.02) == 0,
        u[-1] == 0 , u'[-1] == A,
```

```
      v[-1] == 0, v'[-1] == B  (*initial conditions*)
     },          (*setting governing equations*)
      {u, v}, {x, -1, 1}, PrecisionGoal-> Infinity
    ]
    [[1]]
 );
initialConditions =
 With[          (*contour plotting and root searching*)
    {
     pom = Cases
           [
            Normal
             [
              ContourPlot[Z[A, B], {A, 0, 15}, {B, 0, 15},
              PlotPoints -> 25, Contours -> {0},
              ContourShading -> False]
             ], _Line, Infinity
           ] /. Line[a_] :> a
    },
    Position
     [
      (Differences[#] & /@
       (
        pom /. {a_?NumericQ, b_?NumericQ}:>Sign[R[a, b]]
```

```
        )
       ),
       x_?(# != 0 &)
      ]
      /. {a_Integer, b_Integer} :> pom[[a, b]]
      /. {A_?NumericQ, B_?NumericQ} :>
       FindRoot
       [
        {
         Z[a, b] == 0, R[a, b] == 0
        },
        {a, A}, {b, B}
       ]
    ];
  init[i] = %;          (*store roots in array*)
  i = i + 1;
 ]
For
 [
  j = 1, j < m + 1, j += 1,   (*set up a For loop*)
  L = 10*j;                 (*iterating L*)
  (
   {a, b} /. init[j]) /.
   {a_?NumericQ, b_?NumericQ} :>
```

```
Block
 [{x}, (*declaring local variable x*)
  (
   {u[x], v[x]} /.    (*solving for u[x],v[x]*)
     NDSolve
      [      (*calling solver NDSolve*)
       {
        (p - 1)*(Abs[u'[x]])^(p - 2)*u''[x] +
        ((N - 1)/Abs[x])*u'[x]*(Abs[u'[x]])^(p - 2)+
        L*((Abs[v[x]])^s - 0.01) == 0,
        (q - 1)*(Abs[v'[x]])^(q - 2)*v''[x] +
        ((N - 1)/Abs[x])*v'[x] *(Abs[v'[x]])^(q - 2)+
        L*((Abs[u[x]])^d - 0.02) == 0,
        u[-1] == 0 , u'[-1] == A,
        v[-1] == 0, v'[-1] == B (*initial conditions*)
       },   (*setting governing equations*)
       {u, v}, {x, -1, 1}, PrecisionGoal-> Infinity
      ][[1]]
  )
 ];
% /. {a_, b_} :>
Max
 [
  Table
```

```
        [
         Max[a, b], {x, 0, 1}
        ]
    ];                  (*finding maximum*)

  MM[j] = Max[%]          (*maximum stored in array*)
  ];
ListLinePlot
   [
     {{6, MM[1]}, {8, MM[2]}, ..., {20, MM[7]}},
     AxesLabel -> {L, Subscript \LeftDoubleBracketingBar
    "(u,v)" \RightDoubleBracketingBar, \Infinity},
    PlotRange->{{0, 25}, Automatic}
   ]; (*plotting bifurcation*)
```

APPENDIX D

CODE 4

Laplacian System in an Exterior Domain

```
ClearAll["Global`*"];     (*Clearing variables*)
s = 3;                    (*nonlinearity degree*)
d = 2;                    (*nonlinearity degree*)
zr = 0.0001;              (*declaring near zero*)
m = 10;                   (*setting max iteration*)
Array[init,m];  (*initializing array*)
Array[MM,m];  (*initializing array*)
For
 [
  i = 1, i < 11, i += 1,   (*setting a for loop*)
  L=10*i;
  J[A_?NumericQ, B_?NumericQ] := (*defining function J*)
   (
    u[1] /.     (*defining u[1]*)
     NDSolve
      [      (*calling solver NDSolve*)
       {
        u''[x] + L*(x^(-1/3))*((Abs[v[x]])^s - 0.01)
        == 0,
        v''[x] + L*(x^(-1/2))*((Abs[u[x]])^d - 0.01)
        == 0,
```

```
        u[zr] == 0, v[zr] == 0, u'[zr] == A, v'[zr]
        == B,     (*setting initial conditions*)
        WhenEvent[u[x] < 0, u[1] == -1],
        WhenEvent[v[x] < 0, v[1] == -1]
       },          (*avoiding negative solution*)
        {u,v}, {x, zr, 1}, PrecisionGoal-> Infinity
      ]
   )
   [[1]];
K[A_?NumericQ, B_?NumericQ] := (*defining function J*)
   (
    u[1] /.    (*defining u[1]*)
     NDSolve
      [     (*calling solver NDSolve*)
       {
        u''[x] + L*(x^(-1/3))*((Abs[v[x]])^s - 0.01)
        == 0,
        v''[x] + L*(x^(-1/2))*((Abs[u[x]])^d - 0.01)
        == 0,
        u[zr] == 0, v[zr] == 0, u'[zr] == A, v'[zr]
        == B,     (*setting initial conditions*)
        WhenEvent[u[x] < 0, u[x] -> -1],
        WhenEvent[v[x] < 0, v[x] -> -1]
       },
```

```
        {u,v}, {x, zr, 1}, PrecisionGoal-> Infinity
      ]
  )
  [[1]];
initialConditions =
 With      (*contour plotting and root searching*)
  [
   {
    pom = Cases
           [
            Normal
             [
              ContourPlot[J[A, B], {A, 0, 15}, {B, 0, 15},
              PlotPoints -> 25, Contours -> {0},
              ContourShading -> False]
             ], _Line, Infinity
            ]
            /. Line[a_] :> a
   },
 Position
  [
   (
    Differences[#] & /@
    (
```

```
        pom /. {a_?NumericQ, b_?NumericQ} :>
        Sign[K[a, b]])
       ),
       x_?(# != 0 &)
     ] /. {a_Integer, b_Integer} :> pom[[a, b]] /.
     {A_?NumericQ, B_?NumericQ} :>
     FindRoot
      [
       {
        J[a, b] == 0, K[a, b] == 0
       }, {a, A}, {b, B}
      ]
   ];
   init[i] = %;        (*store roots in array*)
   i = i + 1;
  ]
For
  [
   j = 1, j < m+1, j += 1,   (*set up a For loop*)
   L = 10*j;               (*iterating L*)
   (
    {a, b} /. init[j]) /.
    {a_?NumericQ, b_?NumericQ} :>
      Block
```

```
    [{x}, (*declaring local variable x*)
     (
      {u[x], v[x]} /.    (*solving for u[x],v[x]*)
       NDSolve
         [    (*calling solver NDSolve*)
          {
           u''[x] + L*(x^(-1/3))*((Abs[v[x]])^s - 0.01)
           == 0,
           v''[x] + L*(x^(-1/2))*((Abs[u[x]])^d - 0.01)
           == 0,
           u[zr] == 0, v[zr] == 0, u'[zr] == A, v'[zr]
           == B,  (*setting initial conditions*)
           WhenEvent[u[x] < 0, u[x] -> -1],
           WhenEvent[v[x] < 0, v[x] -> -1]
          },
           {u,v}, {x, zr, 1}, PrecisionGoal-> Infinity
         ]
         [[1]]
     )
    ];
 tt= % /. {a_, b_} :> MaxValue[Max[a, b], {x}];
                                        (*finding maximum*)
MM[j] = Max[tt];  (*maximum stored in array*)
];
```

```
ListLinePlot[Table[{10*i,MM[i],{i,1,m}]]
                        (*plotting bifurcation*)
```